Application of Shakedown Theory in the Design of Earth Structures for Tracks under Repeated Train Loads

Juan Wang[a,b] and Shu Liu [a,b]*

[a] Ningbo Nottingham New Materials Institute, the University of Nottingham Ningbo, Ningbo, China
[b] State Key Laboratory for GeoMechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou, China
* shu.liu@nottingham.edu.cn

Abstract

Tracks require high performance of earth structures against excessive residual settlement under repeated train loads. In the current design standards, good quality of soil materials and compaction control are strictly prescribed based on empirical knowledge and scale tests; however, construction cost becomes relatively high accordingly. A more rational design method based on shakedown theory is presented in this paper. This method makes use of the lower-bound shakedown theorem and models the soils as Mohr-Coulomb materials. By calculating the analytical load-induced dynamic elastic stress field in a three-dimensional half-space and introducing a self-equilibrated residual stress field, the maximum magnitude of surface pressure on the soil foundation (i.e. shakedown limit) against long-term residual settlement is calculated through an optimization program. It is found the shakedown limit is dependent on the ratio of train velocity to shear wave velocity for a given earth structure. The shakedown limits for cases with different train velocities are compared with the design pressure from track analysis.

Keywords: Shakedown, train loads, earth structure, dynamic, lower-bound

1. Introduction

The comfort and safety of the train operation require a smooth and stable trackbed. Especially for high-speed railways, the designs usually require very limited post-construction settlement/differential settlement of the trackbed. Great efforts on material quality and compaction level have been made to try to address this issue. However, it is still found to be very difficult to predict the train-load-induced settlement. On the one hand, the current design codes are established on empirical relations and elastic theory, which may not suit the conditions of different cases and cannot consider the complex nature of the loading condition and the material elastic-plastic responses. On the other hand, researches in this aspect, such as the dynamic interaction among track components under moving load [1-2], the effect of principle stress rotation on soil plastic deformation [3-5], and the physical modelling of the trackbed [6-7], though provide insights into the mechanisms of the load-induced stress and plastic deformation, are very difficult or time-consuming to be applied to different situations in the design practice.

This paper, however, will present a new design method based on a different viewpoint, which aims to find the ultimate status of the trackbed under long-term train loads. The theory of shakedown will be used in this paper, which has been successfully applied in field of pavement engineering to design against excessive rutting [8-16]. It should be noted that, current shakedown design methods for pavements mainly assume a quasi-static response of pavement to traffic loads, which is however not suitable for the design of railways, especially for the high-speed railways. In the railway design standards, the static wheel load is usually multiplied by an amplification factor to include the dynamic effect from the moving train loads. For instance, the Chinese Code for Design High Speed Railway requires the usage of the dynamic amplification factor ($1 + \alpha v$), where v is the velocity of the train (km/h); α is the speed coefficient (= 0.003 for high-speed railways). It is usually considered that the dynamic stresses in soils are amplified by a same factor at a specified train speed. Bian et al. [6] however noted the amplification factor is also dependent on the soil depth based on full-scale physical model test results.

In this paper, shakedown solutions for the railway problem will be presented considering the dynamic effect from the moving train loads. The amplification factor for the soil stresses and the shakedown limits due to the increased train speeds will be investigated. The results can contribute to the design of the earth structures for railway tracks.

2. Shakedown theory

2.1 Notation of shakedown

When an elastoplastic structure is subjected to cyclic loads, the limit provided by limit analysis, beyond which the instantaneous load carrying capacity of the material becomes exhausted, is by no means sufficient to prevent the failure of the material. Although the applied repeated loads may not cause instantaneous collapse of the structure, they possibly induce the plastic deformation in the material in every load cycle and finally results in structural failure in such a way of either alternating plasticity (plastic shakedown) or unlimited incremental plasticity (ratchetting). It is observed in experiments that if the load level is lower than a critical limit, the material will have negligible further development of permanent strains after a number of load cycles and respond elastically to the subsequent load cycles. This phenomenon is termed as 'shakedown' and the critical limit is regarded as the 'shakedown limit'.

The purpose of shakedown analysis is to find the shakedown limit of a structure under cyclic or variable loads. In the field of railway engineering, the shakedown analysis is particularly useful in the prediction of the ultimate status of the trackbed under long-term train loads. That is to say, whether the settlement will keep increasing or reach a stable status.

2.2 Dynamic shakedown theorem

Shakedown analysis can be conducted by using either numerical step-by-step analysis (e.g. [15-16]) or shakedown theorems (e.g. [8-14]). Compared to the costly step-by-step analysis in which the histories of stresses and strains need to be calculated, the methods using the shakedown theorems take their advantages as they can directly obtain the shakedown limit by searching for the critical point or failure mechanism. The pioneering works of the shakedown theorems were conducted by Bleich [17] Melan [18] and Koiter [19] who established the classical static (or lower-bound) and kinematic (or upper-bound) shakedown theorems. Based on the lower-bound shakedown theorem, Ceradni [20] further enunciated and proved the dynamic lower-bound shakedown theorem for elastic perfectly plastic bodies. This theorem states that shakedown will occur in the real response if a fictitious response and a residual stress field may be found so that

$$f(\lambda\sigma_{ij}^{e} + \sigma_{ij}^{r}) \leq 0 \tag{1}$$

where the residual stress field itself σ^{r}_{ij} must satisfy the equilibrium conditions; λ is a dimensionaless factor; the fictitious response refers to the elastic response to the external actions such as the unit load-induced elastic stresses σ^{e}_{ij} and displacements u^{e}_{i}. σ^{e}_{ij} in Eq. (1) should satisfy the following dynamic equilibrium conditions:

$$\sigma_{ij,j}^{e} + X_i = \rho\,\ddot{u}_i^{e} - \chi\dot{u}_i^{e} \tag{2}$$

$$\sigma_{ij}^{e} - f_i = 0 \tag{3}$$

where X_i is the body force field; ρ is the material density; χ is the damping coefficient and f is the surface force. The dynamic lower-bound shakedown theorem can be reduced to Melan's theorem if inertia and damping forces are neglected.

3. Dynamic Shakedown Analysis

3.1 Track analysis

Fig. 1 shows a typical ballastless track system which includes superstructures and supporting earth structures. Four axle loads of 250kN belonging to two adjacent bogies on two carriages are used in the analysis. The loads travel in the direction x at a constant speed v. The rail is UIC60. And the dimensions of the track slab and the concrete base are taken from a typical Rheda 2000 single track system. Properties of each component of the superstructure are summarized in Table 1 according to literature [21]. Assuming the superstructure acts as a single beam with a total EI value and the supporting earth structure is in a relatively good condition with a subgrade modulus of 100 N/cm^3, the displacement of the superstructure can be obtained using the Euler-Bernoulli beam theory. Note the analysis does not consider any dynamic interaction among the components. As a result, the four axle loads will be transformed into a distributed load on the top of the anti-frozen layer as shown in Fig. 2. The maximum vertical stresses on the top of the substructure are comparable with those in full scale physical tests [6] and the pressure suggested in the Hu and Li [21]. Reaction force due to negative displacement is taken as zero. This paper will focus on the effect of the train loads thus does not consider self-weight of the track structure.

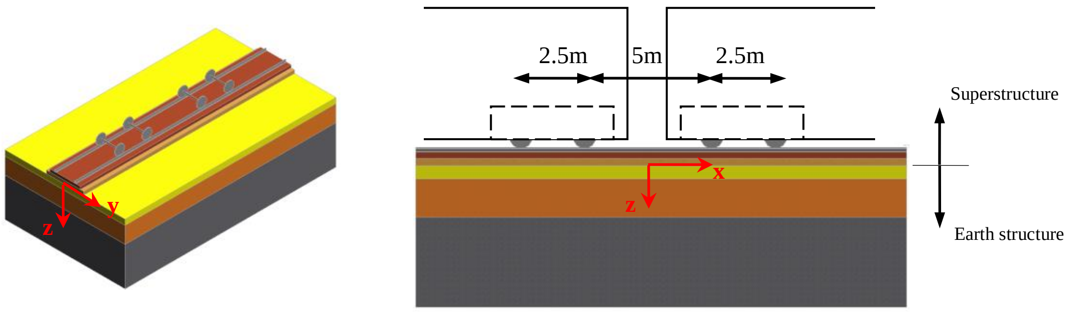
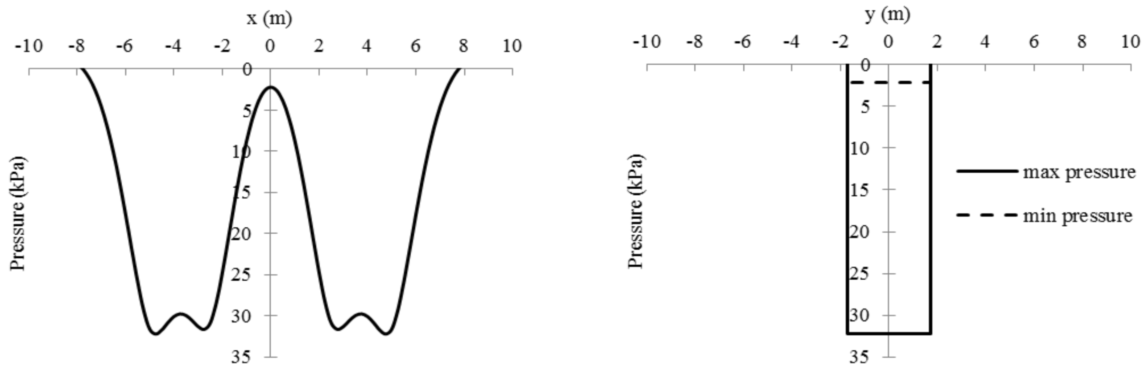


Fig. 1. A typical ballastless track structure and axle loads

Table 1 Material Properties of superstructures

Layer	Young's modulus E (GPa)	Width (cm)	Height (cm)	Second moment of area I (cm^4)
Rail	210	15	17.2	3055
Track slab	34	280	24	322560
Concrete base	10	340	30	765000



(a) Longitudinal direction (b) Transverse direction

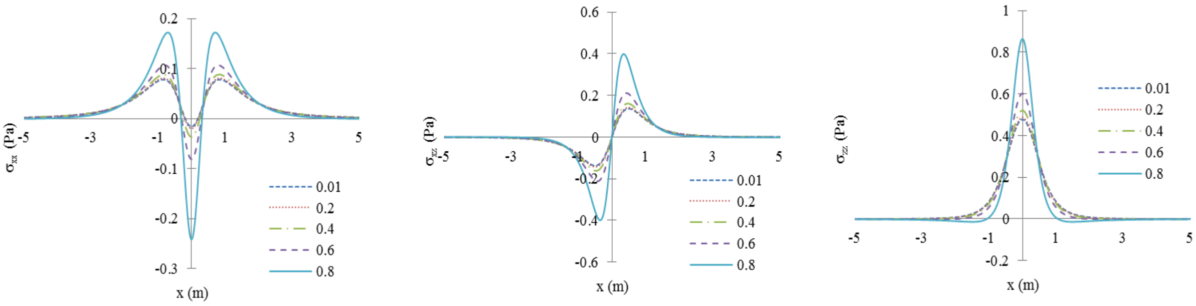
Fig. 2. Pressure on the top of substructure due to wheel loads

3.3 Dynamic elastic stresses

Rather than using an amplification factor for the dynamic soil stresses, analytical solutions of Eason [22] for the stresses produced in a half-space by a moving surface load are used in this paper to obtain the dynamic stresses at a relatively high accuracy. Fig. 3 shows the obtained results for the case of a point load with the change of a velocity factor α_s (defined as v/v_s, v_s is the shear wave velocity). Note compression is positive. It is found that a pronounced tensile stress σ_{xx} occurs directly under the point load. Poisson's ratio also has a slight effect on the stress distribution. In the current study, Poisson's ratio is taken as 0.4.

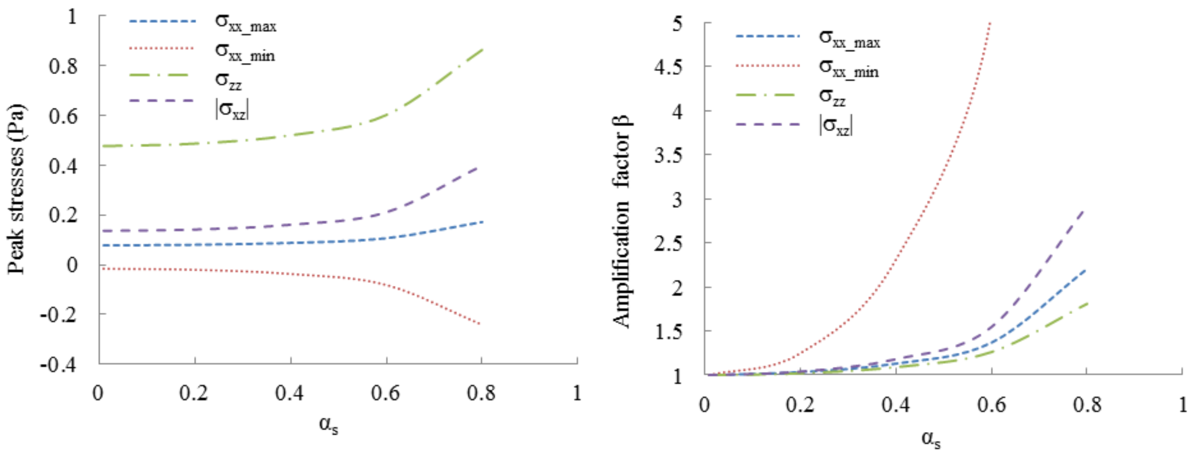
The peak stresses as well as their corresponding amplification factors at the depth z = 1m are also plotted against the velocity factor in Fig. 4. The amplification factor β, defined as the value at the current velocity over that at the quasi-static case (α_s = 0.01), clearly increases with the velocity factor. It is interesting to notice that the vertical dynamic stress σ_{zz}, which is usually used in the railway design, actually grows slowest than the others. It should be noted that the amplification factor does not change with the depth for the case of a point load.

For the train load distribution in Fig 2, the induced dynamic elastic stresses can be calculated by the superposition of the concentrated load solutions. Results for the present study were validated through comparison with references [22-23] and finite element simulations.



(a) (b) (c)

Fig. 3. Stress distributions at z = 1m under a moving unit point load at different values of α_s



(b) (b)

Fig. 4. Effect of α_s on peak stress and amplification factor at z = 1m

3.4 Critical residual stresses and shakedown solutions

According to the dynamic shakedown theorem, the establishment of a residual stress field is essential for the calculation of the shakedown limit. Residual stress is such that can remain in the half-space after the load applications as a result of plastic deformation. For the problem considered here that the half-space retains a flat surface after a number of load passes, every cross-section perpendicular to the travel direction experienced the same load history and therefore the residual stress field is independent of x. Since the pressure is distributed uniformly in the y direction and the length of the pressure is about four times longer than the width, the central plane y = 0 can be considered as the most critical plane. On this plane, the self-equilibrium and boundary conditions eliminate the possibility of σ^r_{xz} and σ^r_{zz}. Therefore, only σ^r_{xx} exists on this plane. The total stresses for a general point on this plane can be defined as the sum of the elastic stresses and the residual stresses. If the total applied load is denoted by λp_{max} (λ is a dimensionless scale parameter, p_{max} is conveniently set as the maximum unit pressure), then all the induced elastic stresses are also proportional to p_{max}, and the total stresses can be expressed as follows:

$$\begin{aligned}\sigma_{xx} &= \lambda\sigma^e_{xx} + \sigma^r_{xx},\\ \sigma_{zz} &= \lambda\sigma^e_{zz},\\ \sigma_{xz} &= \lambda\sigma^e_{xz}.\end{aligned} \tag{4}$$

Assuming the soil material obeys the Mohr-Coulomb yield criterion, the dynamic lower-bound shakedown theorems then requires that the total stress state of any point in the half-space has to lie within the Mohr-Coulomb yield surface. Since σ^r_{yy} can be chosen such that σ_{yy} is an intermediate principle stress, the above requirement leads to the following expression:

$$f = \left(\sigma^r_{xx} + \mathrm{M}\right)^2 + \mathrm{N} \le 0, \tag{5}$$

with

$$\begin{aligned}\mathrm{M} &= \lambda\sigma^e_{xx} - \lambda\sigma^e_{zz} + 2\tan\phi\left(c - \lambda\sigma^e_{zz}\tan\phi\right),\\ \mathrm{N} &= 4\left(1+\tan^2\phi\right)\left[\left(\lambda\sigma^e_{xz}\right)^2 - \left(c - \lambda\sigma^e_{zz}\tan\phi\right)^2\right].\end{aligned}$$

where ϕ and c are the angle of friction and cohesion of the material.

Furthermore, the residual stress σ^r_{xx} at any point i in the half-space must be between two roots of f = 0. For the system to be independent of the travel direction x, the possible residual stress σ^r_{xx} at any depth z is unique and has to lie between two critical residual stresses [9]:

$$\max_x\left(-\mathrm{M}_i - \sqrt{-\mathrm{N}_i}\right) \le \sigma^r_{xx} \le \min_x\left(-\mathrm{M}_i + \sqrt{-\mathrm{N}_i}\right). \tag{6}$$

By substituting either of the critical residual stresses into Equation (5), the shakedown condition of the current problem can be written as:

$$\begin{aligned}&\text{maximise } \lambda,\\ &\text{subject to }\begin{cases} f\left(\sigma^r_{xx}\left(\lambda\sigma^e\right), \lambda\sigma^e\right) \le 0 \\ \sigma^r_{xx}\left(\lambda\sigma^e\right) = \min\limits_{z=j}(-\mathrm{M}_i + \sqrt{-\mathrm{N}_i}) \text{ or } \sigma^r_{xx}\left(\lambda\sigma^e\right) = \max\limits_{z=j}(-\mathrm{M}_i - \sqrt{-\mathrm{N}_i}) \end{cases}.\end{aligned} \tag{7}$$

The above mathematical formulation can be solved by using a program suggested in Wang and Yu [9] for a layered structure thus will not be repeated here. Finally, the maximum admissible load parameter gives the shakedown limit of the half-space $\lambda_{sd}p_{max}$. The shakedown limit is usually represented as a dimensionless factor $\lambda_{sd}p_{max}/c$, known as 'normalized shakedown limit' in the following section.

4. Results and discussion

Analysis is first conducted by assuming identical properties for all soils. In this study, the soils have a stiffness modulus of 275Mpa or 150 MPa, a Poisson's ratio of 0.3 and a density of 2000 kg/m^3. Fig. 5(a) demonstrates the shakedown limit increases with the friction angle but decreases with the velocity. At a specified train speed, a smaller stiffness modulus will lead to a lower shakedown limit. Note that the critical velocity is also a function of the stiffness modulus E, the results are thus further plotted against the velocity factor in Fig. 5(b). As can be seen, the shakedown limits for E=275Mpa and 150Mpa are identical at a certain velocity factor. Consequently the shakedown limits are actually dependent on the velocity factor rather than the value of train velocity. The same to the density. When the train velocity approaches the Rayleigh wave velocity, the shakedown limit drops down to zero.

Fig. 6 presents the amplification factor against the velocity factor. It demonstrates that a higher friction angle leads to a larger amplification factor. The amplification factors are close to those suggested in the design code only when the friction angle is zero and the velocity factor is lower than 0.8. Therefore, though the high friction angle has an positive effect on the long-term stability of railway earth structure, one should be very careful when increasing the train speed at those cases.

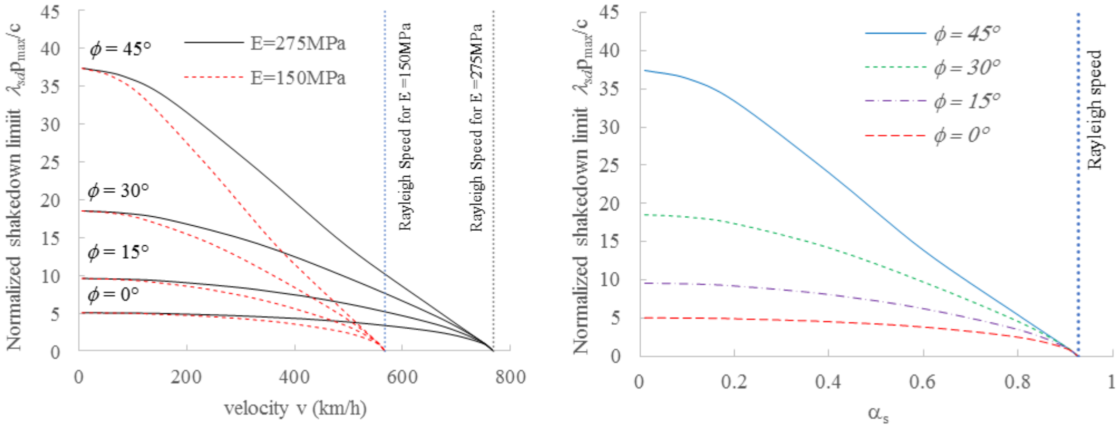


Fig. 5. Effects of ϕ, v, E and α_s on shakedown limit

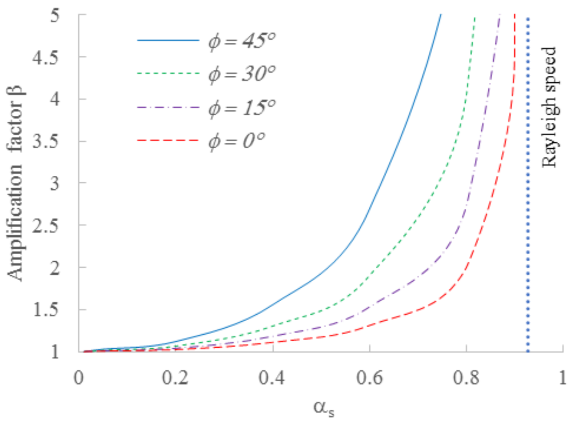


Fig. 6. Effects of ϕ and α_s on amplification factor

Typical earth structures of slab tracks are consisted of several layers of granular materials and soils, namely anti-frozen layer, subgrade bed, embankment fill and subsoil from top to bottom. To solve the layered problem, one common practice is to convert the multi-layered system into an equivalent single layer system. According to Odemark's method, the equivalent thickness of the *n*th layer can be expressed as [24]:

$$h_n^* = h_n \left(\frac{E_n}{E_b} \cdot \frac{1-v_b^2}{1-v_n^2} \right)^{1/3} \tag{8}$$

where h_n is the thickness the *n*th layer; E_n and E_b are the Young's modulus of the *n*th layer and the bottom layer respectively; v_n and v_b are the Poisson's ratio of the *n*th layer and the bottom layer respectively.

Given an earth structure shown in Table 2, the equivalent thickness method could allow the transformation of the layered system into a homogenous system, and therefore the shakedown limit of each equivalent layer can be calculated. Results are shown in Table 2 considering three different train travelling speeds. As can be seen, the shakedown limit decreases with increasing train speed. The critical layer, which provides the lowest layer shakedown limit, changes from the embankment to the anti-frozen layer with rising train speed. The earth structure is stable at a train velocity of 200km/h or lower, but will probably fail in long-term at the velocity of 300km/h (i.e. the shakedown limit is lower the design maximum pressure in Fig. 2). In addition, the shakedown limit is sensitive to the stress distributions in the earth structures; therefore more accurate methods such as finite element method are needed in order to predict the long-term response of the earth structures at a reasonable accuracy. Further investigation will be carried out on this aspect.

Table 2 Shakedown limits of a typical earth structure

Layer	h (m)	h^* (m)	E_d (MPa)	ρ (kg/m^3)	ν	ϕ (°)	c (kPa)	Layer shakedown limit (kPa) v=100	v=200	v=300(km/h)
Anti-frozen layer	0.4	0.5	300	1800	0.3	45	1	47	39	29
Subgrade bed	1.8	2.0	200	1800	0.3	40	2	56	47	35
Embankment	2.8	2.9	175	1800	0.3	30	2	45	39	31
Subsoil	∞	∞	150	1800	0.3	25	2	77	68	55
Final shakedown limit (kPa)								45	39	29

5. Conclusions

In this paper, a procedure to calculate the dynamic shakedown limit for ballastless railways is developed. Dynamic shakedown limits of a typical railway earth structure under a train load at different moving speeds are presented. It is found that the shakedown limit is dependent on the ratio of train velocity to shear wave velocity rather than the train speed only. The increase of the friction angle will largely increase the shakedown limit; however, the amplification factor due to the increase of the train speed from a quasi-static case, is significantly high for the cases with high friction angle. For the layered earth structure, the equivalent thickness method may be used to obtain approximate shakedown limits as a fast approach.

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