Behrouz Emamizadeh and Amin Farjudian*

Monotonicity of the principal eigenvalue related to a non-isotropic vibrating string

Abstract: In this paper we consider a parametric eigenvalue problem related to a vibrating string which is constructed out of *two* different materials. Using elementary analysis we show that the corresponding principal eigenvalue is increasing with respect to the parameter. Using a rearrangement technique we recapture a part of our main result, in case the difference between the densities of the two materials is sufficiently small. Finally, a simple numerical algorithm will be presented which will also provide further insight into the dynamics of the non-principal eigenvalues of the system.

Keywords: Eigenvalue problem, Ordinary differential equation, Principal eigenvalue, Monotonicity, Derivative, Symmetric rearrangements

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Behrouz Emamizadeh: Department of Mathematical Sciences, University of Nottingham Ningbo China, 199 Taikang East Road, Ningbo, 315100, China,
e-mail: Behrouz.Emamizadeh@nottingham.edu.cn
***Corresponding Author: Amin Farjudian:** School of Computer Science, University of Nottingham Ningbo China, 199 Taikang East Road, Ningbo, 315100, China,
e-mail: Amin.Farjudian@nottingham.edu.cn

1 Introduction

The eigenvalue problem associated with a vibrating string, fixed at the ends, is formulated as follows:

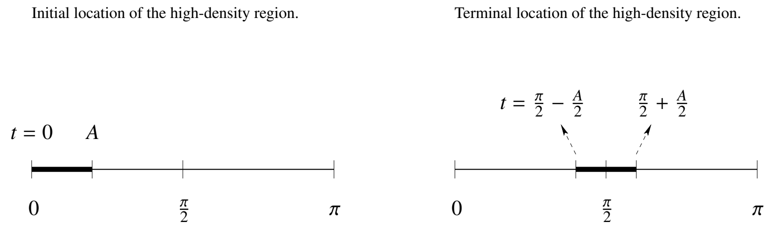
$$\begin{cases} -u'' + f(x)u = \lambda u & \text{in } (a,b) \\ u(a) = u(b) = 0, \end{cases} \tag{1}$$

where $f(x)$ denotes the density of the string. Recall that by a solution to (1) we mean a pair $(\lambda, u) \in \mathbb{R} \times H_0^1(a,b)$ which satisfies the following integral equation:

$$\int_a^b u'v'dx + \int_a^b f(x)uvdx = \lambda \int_a^b uvdx, \quad \forall v \in H_0^1(a,b). \tag{2}$$

If the string is made of N different materials (i. e. the non-isotropic case) with respective non-negative densities $\alpha_1, \ldots, \alpha_N$, then $f(x) = \sum_{j=1}^N \alpha_j \chi_{E_j}$ such that:

Fig. 1 Initial and terminal locations of the high-density region.



(i) Each E_i is a measurable subset of (a, b).
(ii) $\forall i \neq j : \alpha_i \neq \alpha_j$.
(iii) $\forall i \neq j : E_i \cap E_j = \emptyset$.
(iv) $\bigcup_{j=1}^{N} E_j = (a, b)$.

In this case, the differential equation in (1) becomes

$$-u'' + \left(\sum_{j=1}^{N} \alpha_j \chi_{E_j} \right) u = \lambda u,$$

and if we further assume $\alpha_1 = \min \{\alpha_j \mid 1 \leq j \leq N\}$, then we obtain

$$-u'' + \left(\sum_{j=2}^{N} (\alpha_j - \alpha_1) \chi_{E_j} \right) u = (\lambda - \alpha_1) u. \tag{3}$$

In this paper we will focus on the case $N = 2$. Thus, after renaming the coefficients, (3) becomes:

$$-u'' + \alpha \chi_E \, u = \lambda u. \tag{4}$$

More precisely, we will study a particular family of eigenvalue problems of type (4):

$$\begin{cases} -u'' + \alpha \chi_{(t,t+A)}(x) u = \lambda u & \text{in } (0, \pi) \\ u(0) = u(\pi) = 0, \end{cases} \tag{5}$$

where $0 < t \leq \frac{1}{2}(\pi - A)$, and A is a prescribed positive constant such that $A < \pi$. The restriction on the parameter t ensures that the midpoint of the interval $(t, t + A)$ will not exceed $\pi/2$.

In the physical context described at the beginning of this section, equation (5) displays the eigenvalue problem associated with a non-isotropic vibrating string, fixed at the ends, which is constructed out of two different materials. Moreover, the part of the string occupying the region $(t, t + A)$ is made of the material with larger density. As the parameter t moves away from zero and approaches its ultimate value $\frac{1}{2}(\pi - A)$, the region of higher density moves from the far left position towards the middle of the string, as depicted in Figure 1 on the facing page.

It is well known that (5) has infinitely many eigenvalues: $0 < \lambda_1(t) < \lambda_2(t) \leq \lambda_3(t) \leq \cdots \to \infty$. It is the very first one, i. e. $\lambda_1(t)$, called the principal eigenvalue, that is of interest to us. The variational formulation of $\lambda_1(t)$ is as follows:

$$\lambda_1(t) = \inf_{u \in H_0^1(0,\pi),\ \|u\|_2 = 1} \left(\int_0^\pi u'^2\, dx + \alpha \int_0^\pi \chi_{(t,t+A)}(x) u^2\, dx \right). \tag{6}$$

The infimum in (6) is achieved by a unique positive function u_t. The pair $(\lambda_1(t), u_t) \in [0, \infty) \times H_0^1(0, \pi)$ is called the principal eigenpair corresponding to (5). In the particular case of $\alpha = 0$, the principal eigenpair, which is obviously independent of t, turns out to be $(1, \sqrt{2/\pi} \sin x)$. Indeed, the formulation (6), when $\alpha = 0$, confirms $\lambda_1 = 1$. To see this, consider the Fourier sine series of any $u \in H_0^1(0, \pi)$:

$$u(x) = \sum_{n=1}^{\infty} a_n \sin(nx),$$

where

$$a_n = \frac{2}{\pi} \int_0^\pi u(x) \sin(nx)\, dx.$$

On the other hand,

$$u'(x) = \sum_{n=1}^{\infty} n a_n \cos(nx).$$

Whence

$$\int_0^\pi u^2\, dx = \frac{\pi}{2} \sum_{n=1}^{\infty} a_n{}^2 \quad \text{and} \quad \int_0^\pi u'^2\, dx = \frac{\pi}{2} \sum_{n=1}^{\infty} n a_n{}^2.$$

So, $\int_0^\pi u^2 dx \leq \int_0^\pi u'^2 dx$. This, in turn, implies $\lambda_1 \geq 1$. On the other hand, using the test function $v(x) = \sqrt{2/\pi} \sin x$ in (6), we deduce $\lambda_1 \leq 1$. Thus, $\lambda_1 = 1$, as expected.

The main result of this article is the following:

Theorem 1.1. *For $\alpha > 0$, the function $\lambda_1 : (0, \frac{\pi - A}{2}) \to \mathbb{R}$ is strictly increasing.*

The physical interpretation of Theorem 1.1 is that the principal frequency $\lambda_1(t)$ of the string increases as the region with larger density moves from the left end of the interval $(0, \pi)$ towards the middle.

Remark 1.1. *The eigenvalue problem (1) can be interpreted in a different context as well. Indeed, (1) is a scaled version of the one dimensional steady state Schrödinger eigenvalue problem governing a particle of mass m, moving in a potential $V(x)$:*

$$\begin{cases} -(\hbar/2m)u'' + V(x)u = \Lambda u & \text{in } (a,b) \\ u(a) = u(b) = 0, \end{cases}$$

where $\hbar$ denotes the Planck constant. Therefore, the physical interpretation of the assertion in Theorem 1.1, in this new context, is that the principal energy corresponding to the potential $\alpha\chi_{(t,t+A)}$ is strictly increasing as t moves from 0 to $(\pi - A)/2$.

We mention that monotonicity results regarding eigenvalues—and functions of eigenvalues (such as λ_2/λ_1)—of elliptic operators have been extensively investigated in the literature; however, they have been mostly of isoperimetric type, for example, see [AB93a, AB93b, AHS91, Bar85, CO97, Kar98, LY83, LP94, Mar80, Nad95, NP92, Oss78, PPW56, Pól55, Pól61]. There are few papers that address monotonicity of the eigenvalues with respect to a parameter related to the body of the object under study. Some papers investigate the behaviour of the eigenvalues with respect to a parameter which is placed in the boundary conditions, see for example [CGM11] and [LP08]. Our work is primarily motivated by [HLKK01] (also, see [PBN11] and [FBRS08]), where the authors address a problem similar to the one in this note, but in higher dimensions. The advantage of our paper is that the analysis used is elementary and nearly self-contained, hence easily accessible to a wide spectrum of mathematicians and engineers. Theorem 4.3 in Section 4 is somewhat similar to Theorem 2.2 in [EF08], yet bearing a major difference; namely, the maximization problem in [EF08] is performed over a rearrangement class generated by a prescribed positive function whose graph has no *flat* sections. In particular, it cannot be a *characteristic function*, in contrast to the case considered in this note.

In a follow up paper we will generalize Theorem 1.1 in two ways. First, we will prove that the same result holds even if the Dirichlet boundary conditions are replaced with the Robin boundary conditions:

$$\begin{cases} -u'(0) + \gamma_1 u(0) = 0 \\ u'(\pi) + \gamma_2 u(\pi) = 0, \end{cases}$$

where γ_1 and γ_2 are positive constants. Second, we prove that the same result as in Theorem 1.1 can be obtained for the p-Laplacian version of (5):

$$\begin{cases} -(|u|^{p-2}u')' + \alpha\chi_{(t,t+A)}(x)|u|^{p-2}u = \lambda|u|^{p-2}u & \text{in } (0,\pi) \\ u(0) = u(\pi) = 0. \end{cases}$$

However, we are not certain whether the same result holds for the Neumann boundary conditions, see [HLS12] in this regard.

2 Preliminaries

Our first result in this section is an estimate which will prove to be quite useful. Henceforth, C stands for a universal constant whose value may vary from one step to another.

Lemma 2.1. *Consider the boundary value problem:*

$$\begin{cases} -u'' + f(x)u = g(x) & \text{in } (a,b) \\ u(a) = u(b) = 0, \end{cases} \tag{7}$$

where $f \in L^\infty(a,b)$ and $g \in L^2(a,b)$. Then

$$\|u\|_\infty \le C\left(\|f\|_\infty \|u\|_2^2 + \|g\|_2 \|u\|_2\right)^{1/2}. \tag{8}$$

Furthermore, if $f(x)$ is not identically zero, then:

$$\|u\|_\infty \le C\|f\|_\infty^{1/2}\left(\|u\|_2 + \frac{\|g\|_2}{2\|f\|_\infty}\right). \tag{9}$$

Proof. We begin by multiplying the differential equation in (7) by u, and integrating the result over (a,b), to obtain:

$$\|u'\|_2^2 + \int_a^b fu^2\,dx = \int_a^b gu\,dx.$$

Applying the Hölder inequality to the right hand side of the last equation yields:

$$\|u'\|_2^2 + \int_a^b fu^2\,dx \le \|g\|_2\,\|u\|_2,$$

hence,

$$\|u'\|_2^2 \le \|f\|_\infty \|u\|_2^2 + \|g\|_2\,\|u\|_2. \tag{10}$$

On the other hand, since $u(x) = \int_a^x u'(t)\,dt$, we derive:

$$|u(x)| \le \int_a^x |u'(t)|\,dt \le \sqrt{b-a}\,\|u'\|_2, \quad x \in [a,b]. \tag{11}$$

From (10) and (11) we deduce (8). The derivation of (9) from (8) is straightforward. □

Corollary 2.2. *Let $(\lambda_1(t), u_t)$ be the principal eigenpair corresponding to the eigenvalue problem (5). Then:*

(i) $\lambda_1(t) \le C$.

(ii) $\|u_t\|_\infty \le C$.

Proof. Inequality (i) readily follows from Definition (6). To prove (ii), we apply (9), with $f(x) = \alpha\chi_{(t,t+A)}(x)$ and $g(x) = \lambda_1(t)u_t(x)$. Hence, keeping in mind that $\|u_t\|_2 = 1$, we obtain

$$\|u_t\|_\infty \le C\alpha^{1/2}\left(1 + \frac{1}{2\alpha}\lambda_1(t)\right). \tag{12}$$

The inequality (12) coupled with $\lambda_1(t) \le C$ implies (ii). □

Lemma 2.3. *Given $t \in (0,\pi)$, the following estimate holds:*

$$|\lambda_1(t+h) - \lambda_1(t)| \le C|h|, \quad (0 < |h| \ll 1) \tag{13}$$

Proof. Fix a sufficiently small h. From (6), we obtain

$$\begin{aligned}
\lambda_1(t+h) &\le \int_0^\pi u_t'^2\,dx + \alpha\int_0^\pi \chi_{(t+h,t+h+A)}(x)u_t^2\,dx \\
&= \int_0^\pi u_t'^2\,dx + \alpha\int_0^\pi \chi_{(t,t+A)}(x)u_t^2\,dx \\
&+ \alpha\int_0^\pi \left(\chi_{(t+h,t+h+A)}(x) - \chi_{(t,t+A)}(x)\right)u_t^2\,dx \\
&= \lambda_1(t) + \alpha\int_0^\pi \left(\chi_{(t+h,t+h+A)}(x) - \chi_{(t,t+A)}(x)\right)u_t^2\,dx.
\end{aligned} \tag{14}$$

Since $u_t \in L^\infty(0,\pi)$, from (14) we infer

$$\lambda_1(t+h) - \lambda_1(t) \le C|h|. \tag{15}$$

Similarly, one can derive

$$\lambda_1(t) \le \lambda_1(t+h) + \alpha\int_0^\pi \left(\chi_{(t,t+A)}(x) - \chi_{(t+h,t+h+A)}(x)\right)u_{t+h}^2\,dx. \tag{16}$$

From Corollary 2.2, we have $\|u_{t+h}\|_\infty \le C$. Thus, (16) implies

$$\lambda_1(t) - \lambda_1(t+h) \le C|h|. \tag{17}$$

The inequality (13) follows from (15) and (17). □

Lemma 2.4. *Given $t \in (0,\pi)$, the following limit holds:*

$$\lim_{h\to 0}\|u_{t+h} - u_t\|_\infty = 0. \tag{18}$$

Proof. Fix $t \in (0, \pi)$, and consider a numerical sequence (h_n) such that $h_n \to 0$. We will show that (u_{t+h_n}) converges uniformly to u_t, which proves the lemma. To this end, we set $U_n = u_{t+h_n}, \lambda^{(n)} = \lambda_1(t + h_n), I_n = (t + h_n, t + h_n + A)$, and $I = (t, t + A)$. For each n we have:

$$\begin{cases} -U_n'' + \alpha\chi_{I_n}(x)U_n = \lambda^{(n)}U_n & \text{in } (0,\pi) \\ U_n(0) = U_n(\pi) = 0 \end{cases} \tag{19}$$

Multiplying the differential equation in (19) by U_n, integrating the result over $(0, \pi)$, and finally using $\|U_n\|_2 = 1$ we obtain:

$$\|U_n'\|_2^2 + \alpha \int_0^\pi \chi_{I_n}(x)U_n^2\, dx = \lambda^{(n)} \tag{20}$$

Knowing that $\lambda^{(n)} \leq C$, from (20) we infer that (U_n) is bounded in $H_0^1(0, \pi)$. Thus, (U_n) contains a subsequence—still denoted (U_n)—such that $U_n \to U$ weakly in $H_0^1(0, \pi)$, for some $U \in H_0^1(0, \pi)$. Moreover, the same subsequence converges uniformly to U in $(0, \pi)$. Now, we return to (20) and pass n to infinity, keeping in mind that by Lemma 2.3, $\lambda^{(n)} \to \lambda_1(t)$. Hence:

$$\lambda_1(t) \geq \|U'\|_2^2 + \alpha \int_0^\pi \chi_I(x)U^2\, dx. \tag{21}$$

On the other hand, since $\|U\|_2 = 1$, we can apply (6) to deduce that

$$\lambda_1(t) \leq \|U'\|_2^2 + \alpha \int_0^\pi \chi_I(x)U^2\, dx. \tag{22}$$

Therefore, from (21) and (22) we get:

$$\lambda_1(t) = \|U'\|_2^2 + \alpha \int_0^\pi \chi_I(x)U^2\, dx.$$

Whence, by uniqueness of eigenfunctions, we infer that $U = u_t$. □

Lemma 2.5. *The function* $\lambda_1 : (0, \frac{1}{2}(\pi - A)) \to \mathbb{R}$ *is differentiable, and*

$$\lambda_1'(t) = \alpha(u_t^2(t + A) - u_t^2(t)). \tag{23}$$

Proof. We shall show that

$$\lambda_1'(t+) = \alpha(u_t^2(t + A) - u_t^2(t)). \tag{24}$$

To this end, we proceed along the same lines as in the proof of Lemma 2.3. After fixing $0 < h \ll 1$, and setting $I_h = (t+h, t+h+A)$ and $I = (t, t+A)$, one can derive

$$\lambda_1(t+h) \leq \lambda_1(t) + \alpha \int_0^\pi (\chi_{I_h}(x) - \chi_I(x))\, u_t^2 \, dx. \tag{25}$$

From (25), we obtain

$$\frac{\lambda_1(t+h) - \lambda_1(t)}{h} \leq \alpha \int_0^\pi \frac{(\chi_{I_h}(x) - \chi_I(x))}{h} u_t^2 \, dx. \tag{26}$$

Inequality (26) in turn implies

$$\limsup_{h \to 0^+} \frac{\lambda_1(t+h) - \lambda_1(t)}{h} \leq \alpha(u_t^2(t+A) - u_t^2(t)). \tag{27}$$

On the other hand,

$$\lambda_1(t) \leq \lambda_1(t+h) + \alpha \int_0^\pi (\chi_I(x) - \chi_{I_h}(x)) u_{t+h}^2 \, dx.$$

So,

$$\frac{\lambda_1(t) - \lambda_1(t+h)}{h} \leq \alpha \int_0^\pi \frac{(\chi_I(x) - \chi_{I_h}(x))}{h} u_{t+h}^2 \, dx. \tag{28}$$

We will return to (28), but at this point we shall show:

$$\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} u_{t+h}^2 \, dx = u_t^2(t). \tag{29}$$

To this end,

$$\frac{1}{h} \int_t^{t+h} u_{t+h}^2 \, dx - u_t^2(t) = \frac{1}{h} \int_t^{t+h} (u_{t+h}^2(x) - u_t^2(x)) dx + \frac{1}{h} \int_t^{t+h} (u_t^2(x) - u_t^2(t)) dx. \tag{30}$$

From Lemma 2.4 we infer that the first integral on the right hand side of (30) tends to zero as h tends to zero. The second integral clearly tends to zero as well. This finishes the proof of our claim (29). Similarly, one can show:

$$\lim_{h \to 0^+} \frac{1}{h} \int_{t+A}^{t+h+A} u_{t+h}^2 \, dx = u_t^2(t+A). \tag{31}$$

In view of (29) and (31), we infer from (28):

$$\limsup_{h\to 0^+} \frac{\lambda_1(t) - \lambda_1(t+h)}{h} \le \alpha(u_t^2(t) - u_t^2(t+A)). \tag{32}$$

From (32) and (27) we deduce (24).

Using similar arguments as above, one can also show:

$$\lambda_1'(t-) = \alpha(u_t^2(t+A) - u_t^2(t)). \tag{33}$$

The combination of (24) and (33) implies (23). □

3 Proof of Theorem 1.1

Proof. (Theorem 1.1) In view of Lemma 2.5, we need to show:

$$u_t(t+A) > u_t(t), \quad \forall t \in (0, (\pi - A)/2).$$

To this end, we fix $t \in (0, (\pi - A)/2)$, and let m denote the midpoint of the interval $I(t) = (t, t+A)$. For $x \in (0, m)$, let x^m denote the reflection of x relative to m, i. e. $x^m = 2m - x$. Next, we introduce the function:

$$w(x) = u_t(x) - u_t(x^m), \quad x \in (0, m).$$

Note that if we show $w(t) < 0$, then we are done. As we shall see, a stronger result will be proved; namely, w is negative in the entire interval $(0, m)$. We prove this claim in two steps. First we show that w is non-positive on its domain, and this, in turn, paves the path toward the second step which is the application of the strong maximum principle to draw the conclusion that w is in fact negative in $(0, m)$.

Let us observe that:

$$\begin{aligned} -w''(x) + \alpha\chi_{I(t)}(x)w(x) &= -(u_t''(x) - u_t''(x^m)) + \alpha\chi_{I(t)}(x)(u_t(x) - u_t(x^m)) \\ &= -u_t''(x) + \alpha\chi_{I(t)}(x)u_t(x) - (-u_t''(x^m) + \alpha\chi_{I(t)}(x^m)u_t(x^m)) \\ &= \lambda_1(t)u_t(x) - \lambda_1(t)u_t(x^m) = \lambda_1(t)w(x), \end{aligned}$$

since $\chi_{I(t)}(x) = \chi_{I(t)}(x^m)$. Moreover, $w(m) = 0$ and $w(0) < 0$, since $u(0) = 0$ and $u(0^m) = u(2m) > 0$. Therefore, we have:

$$\begin{cases} -w'' + \alpha\chi_{I(t)}(x)w = \lambda_1(t)w & \text{in } (0, m) \\ w(0) = w_0 < 0, \ w(m) = 0. \end{cases} \tag{34}$$

To show w is non-positive, it suffices to show that $w^+ := \max\{0, w(x)\}$ is identically zero in $(0, m)$. Clearly, $w^+ \in H_0^1(0, m)$, hence, from (34), we deduce:

$$\int_0^m {w^+}'^2 \, dx + \alpha \int_0^m \chi_{I(t)}(x) {w^+}^2 \, dx = \lambda_1(t) \int_0^m {w^+}^2 \, dx. \tag{35}$$

From (35) we infer $\lambda_1(t) \geq \tilde{\lambda}$, where $\tilde{\lambda}$ is the principal eigenvalue of the following problem:

$$\begin{cases} -Z'' + \alpha\chi_{I(t)}(x)Z = \lambda Z & \text{in } (0, m) \\ Z(0) = Z(m) = 0. \end{cases}$$

However, $\lambda_1(t) < \tilde{\lambda}$, which follows from the variational formulation of $\lambda_1(t)$, noting that $H_0^1(0, m)$ is trivially embedded into $H_0^1(0, \pi)$, by extending the elements of $H_0^1(0, m)$ to be zero in $(0, \pi) \setminus (0, m)$. So, we derive a contradiction: $\tilde{\lambda} \leq \lambda_1(t) < \tilde{\lambda}$. Whence, w^+ is identically zero in $(0, m)$, as desired. This proves w is non-positive. Finally, the strong maximum principle applied to the boundary value problem (34) implies w is negative in $(0, m)$, which completes the proof of the theorem. □

4 Further discussion

From Theorem 1.1, we infer $\lambda_1(t) < \lambda_1((\pi - A)/2)$, for all $0 < t < (\pi - A)/2$. In this section we discuss how this result can be obtained using rearrangement inequalities, provided that α is small enough. We start with the following:

Lemma 4.1. *Let (λ_1, u) denote the principal eigenpair for the eigenvalue problem:*

$$\begin{cases} -S'' + \alpha\chi_I(x)S = \lambda S & \text{in } (0, \pi) \\ S(0) = S(\pi) = 0, \end{cases} \tag{36}$$

where $I = ((\pi - A)/2, (\pi + A)/2)$. Then, $\forall x \in (0, \pi) : u(x) = u(\pi - x)$.

Proof. Define $w(x) = u(x) - u(\pi - x)$. Then

$$\begin{aligned} -w''(x) + \alpha\chi_I(x)w(x) &= -u''(x) + \alpha\chi_I(x)u(x) \\ &\quad - (-u''(\pi - x) + \alpha\chi_I(\pi - x)u(\pi - x)) \\ &= \lambda_1 u(x) - \lambda_1 u(\pi - x) = \lambda_1 w(x). \end{aligned}$$

Moreover, $w(0) = w(\pi) = 0$. Hence, it follows that either w is identically zero or it is a multiple of u. Let us assume the latter is true, i. e. $w = \beta u$, for some non-zero constant β. Whence, w is either positive or negative, which in either case contradicts the fact that $w(x) = -w(\pi - x)$. Thus, $w \equiv 0$, and the assertion of the lemma is proved. □

Lemma 4.2. *There exists $\overline{\alpha} > 0$ such that if $0 < \alpha < \overline{\alpha}$ and $(\lambda_\alpha, u_\alpha)$ is the eigenpair for the eigenvalue problem (36), then u_α is strictly decreasing in $(\pi/2, \pi)$.*

Proof. Define $\xi(\alpha) = \lambda_\alpha - \alpha$. We show that ξ is strictly decreasing on $[0, \infty)$, using the ideas introduced in [CGI+00]. To this end, we consider $0 < \alpha' < \alpha$, and for simplicity set $u = u_\alpha$ and $v = u_{\alpha'}$. Thus,

$$\begin{aligned}
\xi(\alpha) = \lambda_\alpha - \alpha \quad &\leq \quad \int_0^\pi v'^2 dx + \alpha \int_0^\pi \chi_I(x) v^2 \, dx - \alpha \\
&= \quad \int_0^\pi v'^2 dx + \alpha' \int_0^\pi \chi_I(x) v^2 \, dx + (\alpha - \alpha') \int_0^\pi \chi_I(x) v^2 \, dx - \alpha \\
&= \quad \lambda_{\alpha'} - \alpha' + (\alpha' - \alpha) \int_0^\pi (1 - \chi_I(x)) v^2 \, dx \qquad (37) \\
&< \quad \lambda_{\alpha'} - \alpha' = \xi(\alpha'),
\end{aligned}$$

since $\int_0^\pi v^2 \, dx = 1$. So, ξ is strictly decreasing, as desired. Observe that (37) implies $\lim_{\alpha \to \infty} \xi(\alpha) = -\infty$. This, coupled with the fact that $\xi(0) = \lambda_0 = 1$, ensure existence of a unique $\overline{\alpha}$ such that $\xi(\overline{\alpha}) = 0$.

Henceforth, we assume $\alpha < \overline{\alpha}$. Let us recall the differential equation satisfied by u:

$$-u'' + \alpha \chi_I(x) u = \lambda_\alpha u \quad \text{in } (0, \pi). \tag{38}$$

By Lemma 4.1, $u(x) = u(\pi - x)$ in $(0, \pi)$, hence $u'(\pi/2) = 0$. Therefore, from (38), we obtain:

$$-\int_{\frac{\pi}{2}}^{x} u''(y) \, dy = \int_{\frac{\pi}{2}}^{x} (\lambda_\alpha - \alpha \chi_I(x)) u \, dy, \quad \forall x \in (\pi/2, \pi). \tag{39}$$

From (39), we infer

$$-u'(x) \geq \int_{\frac{\pi}{2}}^{x} (\lambda_\alpha - \alpha) u \, dy = \int_{\frac{\pi}{2}}^{x} \xi(\alpha) u \, dy > 0, \quad \forall x \in (\pi/2, \pi),$$

since $\xi(\alpha) > 0$. So, the proof of the lemma is completed. □

The main result of this section is the following:

Theorem 4.3. *Let $t \in (0, (\pi - A)/2)$, and $\alpha < \overline{\alpha}$. Then $\lambda_\alpha(t) \leq \lambda_\alpha((\pi - A)/2)$.*

Proof. For fixed $t \in (0, (\pi - A)/2)$, we set $\lambda = \lambda_\alpha(t)$ and $\overline{\lambda} = \lambda((\pi - A)/2)$. Also, we assume (λ, u) and $(\overline{\lambda}, v)$ are principal eigenpairs. Then

$$\lambda \leq \int_0^\pi v'^2 \, dx + \alpha \int_0^\pi \chi_{I(t)}(x) v^2 \, dx, \tag{40}$$

where $I(t) = (t, t + A)$. At this point we apply the Hardy-Littlewood rearrangement inequality (see for example [Kaw85]) to obtain:

$$\int_0^\pi \chi_{I(t)}(x) v^2 \, dx \leq \int_0^\pi \left(\chi_{I(t)}\right)^* (x) v^{*2} \, dx, \tag{41}$$

where $(.)^*$ denotes the well known symmetric rearrangement operator relative to the line $x = \frac{\pi}{2}$ in the xy-plane. On the other hand from Lemma 4.1 and Lemma 4.2, we have $v^* = v$, so from (40) and (41), we obtain:

$$\lambda \leq \int_0^\pi v'^2 \, dx + \alpha \int_0^\pi \left(\chi_{I(t)}\right)^* (x) v^{*2} \, dx = \int_0^\pi v'^2 \, dx + \alpha \int_0^\pi \chi_I(x) v^2 \, dx = \overline{\lambda},$$

where $I = ((\pi - A)/2, (\pi + A)/2)$. So, the proof of the theorem is completed. □

5 Numerical simulation

The eigenvalue problem (1) can be solved numerically in various ways. Here we briefly describe a simple ansatz based on Galerkin's method which we have used in our numerical algorithm. A succinct presentation of the underlying approach may be found in [Eds08].

5.1 Reduction to a generalized eigenvalue problem

Consider the formulation (2) on page 1 in which $f(x)$ is replaced with $\alpha\chi_{(t,t+A)}(x)$, i. e.

$$\int_a^b u'v'dx + \int_a^b \alpha\chi_{(t,t+A)} \, uvdx = \lambda \int_a^b uvdx, \quad \forall v \in H_0^1(a, b) \tag{42}$$

Assume that $N \geq 1$ and set $h = (b-a)/(N+1)$. Next, for each $1 \leq i \leq N$, let the *roof* function ϕ_i^N be defined by

$$\forall x \in (a,b) : \phi_i^N(x) = \begin{cases} (x-a-(i-1)h)\,/\,h & \text{if} \quad a+(i-1)h < x < a+ih \\ (a+(i+1)h-x)\,/\,h & \text{if} \quad a+ih \leq x < a+(i+1)h \\ 0 & \text{otherwise} \end{cases}$$

Note that the collection $\{\phi_i^N \mid 1 \leq N, 1 \leq i \leq N\}$ forms a basis for the Sobolev space $H_0^1(a,b)$. We drop the superscript N and simply write ϕ_i where there is no confusion. In order to solve equation (42), we approximate u by an ansatz $\tilde{u}$ satisfying

$$\tilde{u}(x) = \sum_{i=1}^{N} v_i \phi_i(x) \tag{43}$$

in which the coefficients $\{v_i \mid 1 \leq i \leq N\}$ are unknown. Knowing that equation (42) holds for all $v \in H_0^1(a,b)$, by substituting ϕ_j for v for each $1 \leq j \leq N$, one obtains:

$$\int_a^b \tilde{u}'\,\phi_j'\,dx + \int_a^b \alpha\chi_{(t,t+A)}\,\tilde{u}\,\phi_j\,dx = \lambda \int_a^b \tilde{u}\,\phi_j\,dx \tag{44}$$

Next we need to discretise the characteristic function $\chi_{(t,t+A)}$. Define the set $\mathcal{A} := \{a+ih \mid 0 < i < N+1\}$ and let t_0 and t_1 be the smallest and the largest elements of $\mathcal{A}$ that lie in the interval $(t, t+A)$, respectively. We approximate the characteristic function $\chi_{(t,t+A)}$ by $\tilde{\chi} := \chi_{(t_0,t_1)}$.

By using (43), incorporating $\tilde{\chi}$, and then rearranging terms in (44), one gets

$$\sum_{i=1}^{N}\left(\int_a^b \phi_i'\,\phi_j'\,dx\right)v_i + \sum_{i=1}^{N}\left(\alpha\int_a^b \tilde{\chi}\,\phi_i\,\phi_j\,dx\right)v_i = \lambda\sum_{i=1}^{N}\left(\int_a^b \phi_i\,\phi_j\,dx\right)v_i$$

which reduces to

$$\sum_{i=1}^{N}\left(\int_a^b \phi_i'\,\phi_j'\,dx + \alpha\int_a^b \tilde{\chi}\,\phi_i\,\phi_j\,dx\right)v_i = \lambda\sum_{i=1}^{N}\left(\int_a^b \phi_i\,\phi_j\,dx\right)v_i$$

Let us consider the $N \times N$ matrices C, D and M whose entries are as follows:

$$\forall i,j \in \{1,\dots N\} : \begin{cases} C_{i,j} &= \int_a^b \phi_i'\,\phi_j'\,dx \\ D_{i,j} &= \alpha\int_a^b \tilde{\chi}\,\phi_i\,\phi_j\,dx \\ M_{i,j} &= \int_a^b \phi_i\,\phi_j\,dx \end{cases}$$

Some straightforward calculations would reveal that C, D and M are all tridiagonal matrices of the following form:

$$C = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}, \quad M = \frac{h}{6}\begin{bmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 4 & 1 \\ 0 & \cdots & 0 & 1 & 4 \end{bmatrix}$$

and due to the presence of $\tilde{\chi}$, the matrix D is a 'cropped' version of αM, i. e.

$$D_{i,j} = \begin{cases} \alpha M_{i,j} & \text{if } \frac{t-a}{b-a} < \frac{i}{N+1} < \frac{t+A-a}{b-a} \\ 0 & \text{otherwise} \end{cases}$$

Hence, we have obtained the generalised eigenvalue problem :

$$(C + D) \times V = M \times V \times \Lambda \tag{45}$$

in which:
– V is an $N \times N$ matrix whose columns form the eigenvectors. The entries in each column can be substituted in (43) to obtain various approximations of u.
– Λ is a diagonal matrix containing the eigenvalues on its diagonal.

The problem (45) can be solved using any of the established methods for solving a generalised eigenvalue problem. Note that in order to obtain a simple eigenvalue problem both sides of (45) need to be multiplied by M^{-1}. However, one should refrain from that extra step as it leads to full matrices which in turn would add significantly to the cost of representations and computations. A well designed algorithm for solving generalised eigenvalue problems can take advantage of the tridiagonal structure of the matrices involved.

5.2 Accuracy and convergence

How accurate a solution one would obtain from (45) depends on the value of N. It is best to start off with an initial N_0, and then try the algorithm with successive values N_i until some measure of convergence is observed. For instance, assume that:

$$a = 0, \quad b = \pi, \quad t = \frac{9\pi}{20}, \quad A = \frac{\pi}{10}$$

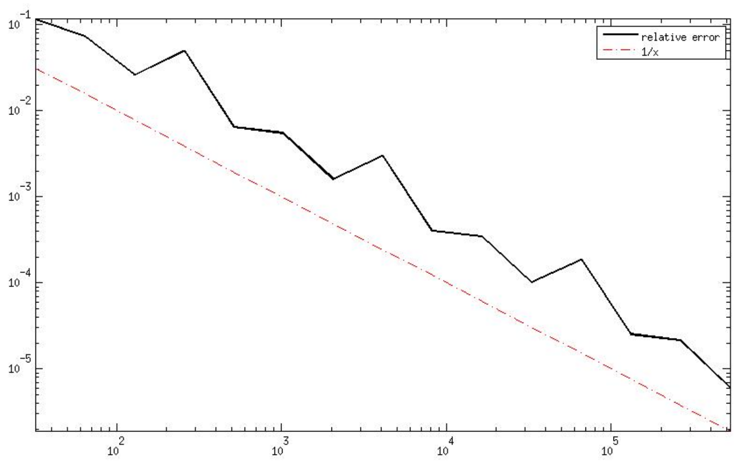
We start off with $N_0 = 4$ and continue by doubling the values according to $N_{i+1} = 2N_i$. At each iteration i we calculate the relative error

$$e_i = \left| \frac{\lambda_1^{N_{i+1}} - \lambda_1^{N_i}}{\lambda_1^{N_i}} \right|$$

in which $\lambda_1^{N_i}$ denotes the smallest eigenvalue of system (45) with parameter N_i.

We ran the algorithm until the relative error went below 10^{-5}, for which the graph of the relative error with respect to N is shown in Figure 2 below. As can be seen from the figure, the convergence is quite fast.

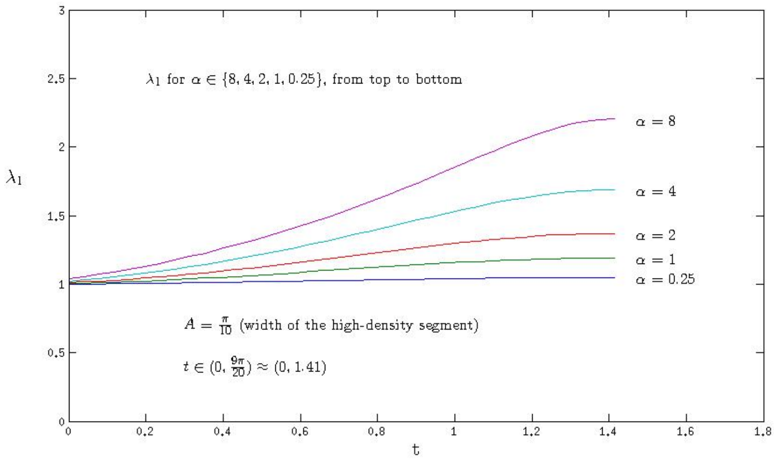
Fig. 2 Relative error of successive approximations to λ_1 with respect to N. The plot of $1/x$ is shown in dashed line for convenient comparison. Note that the plots are logarithmic along both x and y axes.



5.3 Different values for α

We used the underlying set up described so far to produce the graph of $\lambda_1(t)$ for different values of α, as shown in Figure 3 on the next page. Note that according to this figure, $\lambda_1(t)$ approaches 1 as α tends to zero. One can prove that this convergence is uniform. Indeed, since $\inf_{u\in H_0^1(0,\pi),\, \|u\|_2=1} \int_0^\pi u'^2 dx = 1$, it is clear from (6) on page 3 that $1 \leq \lambda_1(t) \leq 1 + \alpha$, for all t in $(0, \pi)$. Therefore $\lambda_1(t) = 1 + O(\alpha)$, as $\alpha \to 0^+$, uniformly in t.

Fig. 3 The values of $\lambda_1(t)$ tend to 1 (for all $t \in (0, \pi)$) as $\alpha \to 0$.



5.4 Other eigenvalues

One of the benefits of numerical simulations is that they can provide us with some insight into aspects of a system before the theory is developed. A case in point is the dynamics of the other eigenvalues besides λ_1, even though in the current paper we have focused exclusively on the principal eigenvalue.

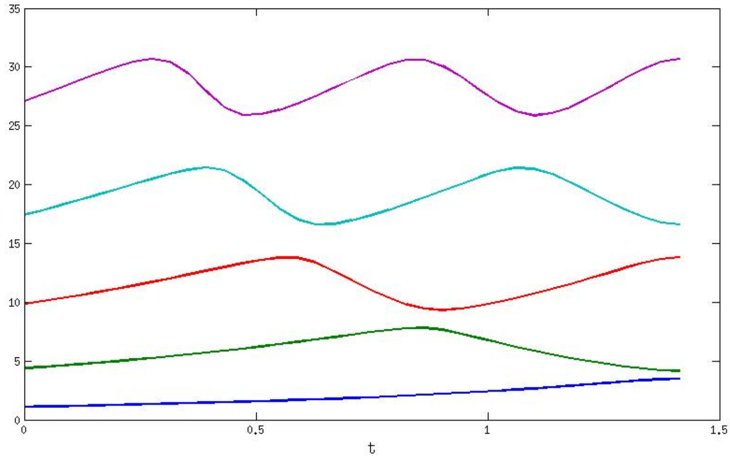
A plot of the first five eigenvalues are shown in Figure 4 on the next page. It seems that although λ_1 grows monotonically as the high density segment moves towards the center, λ_2 increases towards a peak and then goes down. In other words, in $(0, (\pi-A)/2)$, λ_2 goes through one local optima. The eigenvalue λ_3 goes through a local maximum and then a local minimum, i. e. the plot of λ_3 has 2 local optima.

It appears from the figure that in $(0, (\pi - A)/2)$ each λ_k goes through $(k - 1)$ local optima.

6 Conclusion

In this note we considered an eigenvalue problem related to a non-isotropic vibrating string which is fixed at the two ends. Notably, the string is made of two different materials. We showed that as the location of the material with larger density moves continuously from either left or right ends toward the middle of the string, the corresponding principal eigenvalue increases. We also used the Hardy-Littlewood rearrangement inequality to show that the principal eigenvalue—in case the location of the string bearing

Fig. 4 The first five eigenvalues (i. e. λ_1 through λ_5) from bottom to top. It seems that each λ_k has $(k-1)$ critical points in $(0, (\pi - A)/2)$.



larger density is precisely in the middle—exceeds the principal eigenvalue corresponding to the density distribution where the location of the material with larger density is closer to the ends. It will be interesting to know if the results of Theorem 1.1 and Theorem 4.3 still hold if the Dirichlet boundary conditions in (1) are replaced with the Neumann boundary conditions.

In a follow up paper we will show that an analysis similar to the one presented in this paper can be applied to the eigenvalue problem:

$$\begin{cases} -(|u|^{p-2}u')' + \alpha\chi_{(t,t+A)}(x)|u|^{p-2}u = \lambda|u|^{p-2}u & \text{in } (0,\pi) \\ u(0) = u(\pi) = 0. \end{cases}$$

and that the same result as in Theorem 1.1 will hold. We will also prove that the same result still stands even if the Dirichlet boundary conditions are replaced with the Robin boundary conditions:

$$\begin{cases} -u'(0) + \gamma_1 u(0) = 0 \\ u'(\pi) + \gamma_2 u(\pi) = 0, \end{cases}$$

where γ_1 and γ_2 are positive constants. Whether the result is valid under Neumann boundary conditions remains to be investigated.

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