Global Existence and Blow-Up Phenomena in Keller-Segel-Type Systems with Tensorial Flux

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Abstract

In the past decade, there has been much interest in analyzing Keller-Segel models with tensorial flux. However, it is not yet well understood whether there are solutions that blow-up in a finite amount of time. This thesis aims to bridge this gap by developing a comprehensive approach capable of yielding sharp results regarding global existence and blow-up phenomena across various systems characterized by the interplay between vorticity and one or more chemotactic signals. Furthermore, significant progress has been made in resolving numerous open problems pertaining to the existence of solutions for diverse mathematical models in the realms of mathematical physics and biology, cf. [82, 87, 92, 93]. Moreover, Significant advancements have been made in analyzing Keller-Segel models with tensorial flux in both two and higher dimensions, achieved through the introduction of a novel technique. This technique showcases the possibility of finite-time blowup solutions in the Keller-Segel model, even under highly general conditions on the tensorial flux. On the other hand, cells encounter a diverse array of physical and chemical signals as they navigate their natural surroundings. However, their response to the simultaneous presence of multiple cues remains elusive. Particularly, the impact of topography alongside a chemotactic gradient on cell migratory behavior remains insufficiently explored. So, it is noteworthy that among the innovations of this thesis, we also delve into analyzing the conditions that predict or prevent cell aggregation when obstacles interfere during the process.

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Contents

Chapter 1

Introduction

Over the past decade, significant attention has been directed towards analyzing Keller-Segel models incorporating tensorial flux. While numerous papers have delved into the scientific literature, elucidating enough conditions for global existence, the qualitative dynamics of these models remain inadequately comprehended. Specifically, the inquiry into whether solutions exhibit finite-time blow-up remains a lingering question in the general case. Thus, despite strides in establishing global existence criteria, a comprehensive understanding of the qualitative behavior of Keller-Segel models with tensorial flux continues to elude researchers.

To provide the essential background and highlight the relevance of this research, it is necessary to revisit key definitions and mathematical results of the models under scrutiny.

Initially, it is imperative to introduce the fundamental concept surrounding the mathematical description of organismal response to a chemical stimulus, known in biological contexts as chemotaxis. It can be categorized into positive chemotaxis, where movement occurs towards higher concentrations of the chemical, and negative chemotaxis, where movement is directed away from the chemical gradient. This topic holds significant prominence in contemporary research literature, as it is pivotal for understanding various biological processes. Examples include the spontaneous aggregation of the amoeba Dictyostelium Discoideum to locate food sources [52], coral fertilization [53], embryonic development [70], and the spreading of cancer cells [21] , among others.

The fundamental features of cell aggregation through chemotaxis can be described in the two-dimensional case by the next version of the classical Keller–Segel model [52]

$$\begin{aligned} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) \quad & x \in \mathbb{R}^2, t > 0, \\ -\Delta v = u \quad & x \in \mathbb{R}^2, t > 0, \\ u(x,0) = u_0(x) \quad & x \in \mathbb{R}^2. \end{aligned} \tag{1.1}$$

Here u denotes the density of cells and v represents the concentration of the chemoattractant. When $\chi > 0$ the motion is in direction to the gradient of concentration of v. The case $\chi < 0$ describes the motion in the opposite direction of the gradient of concentration of v.

When $\chi > 0$ the cells aggregate. In this case, it is natural to ask whether the solution of the Keller-Segel model blows up. A conclusive answer regarding the blow-up phenomenon was provided for the model (1.1) in 2004 (e.g. [32]) showing that in the case $0 \leq \int_{\mathbb{R}^2} u_0(x)dx < \frac{8\pi}{\chi}$ the solution exists globally in time, meanwhile if $\int_{\mathbb{R}^2} u_0(x)dx > \frac{8\pi}{\chi}$ a blow-up is possible. This threshold phenomenon for the model (1.1) has brought the attention of the research community and thousands of papers analyzing the qualitative behavior of the Keller-Segel model have been published, cf. [3].

An interesting variation of model (1.1) arise when taking into account that chemotactic migration in certain situations, is not necessarily parallel to the gradient of the signal. An example is given by the dynamics of a type of bacteria known as peritrichously flagellated when swimming close to surfaces (e.g. [31, 92, 93]). In this case the evolution of the density of bacteria is described in two dimensional case by

$$u_t = \Delta u - \nabla \cdot (uA(x,u,v)\nabla v), \ \ x \in \mathbb{R}^2, t > 0,$$

where the symbol $A(x,u,v)$ represents a 2×2 matrix, which makes quite challenging or even impossible the application of the standard techniques, known for the case $A = I$, to find the conditions for having either blow-up or global existence of solutions. To the best of our knowledge, there are two main open problems related to this system:

1. It has not been found a full description of the structure of the matrix A allowing to conclude the existence of global solutions. However, it is worth to mentioning that there are several partial results for similar Keller-Segel type systems when A satisfies quite restricted conditions of decayment; see for instance [58, 84, 86, 87] and the references therein.

2. The problem of showing the existence of solutions blowing up in a finite time remains open in the general case.

A meaningful result in this direction, was recently reported for the parabolic-elliptic model

$$\begin{array}{ll} u_t = \Delta u - \chi \nabla \cdot (uA_\alpha \nabla v) & x \in \mathbb{R}^2, t > 0, \\ -\Delta v = u & x \in \mathbb{R}^2, t > 0, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^2, \end{array} \tag{1.2}$$

where χ represents a positive constant and

$$A_a := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{1.3}$$

denotes a rotation matrix with $\alpha \in (-\pi, \pi]$ constant. Namely, the authors in [38] proved that If $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and the initial data u_0 satisfies $0 \leq \int_{\mathbb{R}^2} u_0(x)dx < \frac{8\pi}{\chi \cos\alpha}$, then the solution exists globally in time, meanwhile the condition $\int_{\mathbb{R}^2} u_0(x)dx > \frac{8\pi}{x \cos\alpha}$ implies the possibility of having blow-up. Meanwhile it was shown that in the case $\alpha \in \left(-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]$ the solution exists globally in time independently on the value of $\int_{\mathbb{R}^2} u_0(x)dx$.

Note that, by taking different values of α in (1.3), we recover several mathematical models arising in mathematical biology. The simplest case occurs when $\alpha = 0$, and therefore the matrix A_α becomes the identity matrix recovering the classical Keller-Segel model (1.1). In the case $\alpha = \pi$, we have that $A_\alpha = -I$, and therefore we rescue the Keller-Segel model with negative chemotaxis

$$\begin{aligned} u_t &= \Delta u + \chi \nabla \cdot (u \nabla v) & x \in \mathbb{R}^2, t > 0, \\ -\Delta v &= u & x \in \mathbb{R}^2, t > 0, \\ u(x,0) &= u_0(x) & x \in \mathbb{R}^2. \end{aligned}$$

When $\alpha = -\pi/2$, we get

$$A_\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and, by taking $\chi = 1$ we obtain the vorticity equation for a two-dimensional flow (cf. [45, Section 2.1.])

$$\begin{aligned} u_t &= \Delta u + \nabla \cdot (u \nabla^\perp v), & x \in \mathbb{R}^2, t > 0, \\ -\Delta v &= u, & x \in \mathbb{R}^2, t > 0, \\ u(x,0) &= u_0(x), & x \in \mathbb{R}^2, \end{aligned} \tag{1.4}$$

where the differential operator $\nabla^\perp$ is called the anti-gradient operator, and is defined by

$$\nabla^\perp v := \begin{pmatrix} -\partial_{x_2} v \\ \partial_{x_1} v \end{pmatrix}.$$

The vorticity is a fundamental physical quantity in fluid dynamics, used to quantify the rotation of a fluid. It serves as a crucial tool in understanding various atmospheric and oceanic flows. The vorticity equation (1.4) can be derived from the Navier-Stokes equations by applying the curl differential operator, cf. [45, Section 2.1.].

In this regard, the significant contribution in [38] lies in its revelation that several important systems of partial differential equations found in mathematical biology, fluid dynamics, and electrokinetics can be encompassed within a unified framework. Specifically, the authors introduce a Keller-Segel-type system with rotational flux terms. Nevertheless, there remains a significant amount of work to be done in the analysis of Keller-Segel models that incorporate tensorial flux.

In Chapter 2, a more detailed literature review delves deeper into the Keller-Segel model, offering an analysis of its historical development, theoretical underpinnings, and notable advancements. By synthesizing a wide array of scholarly contributions, this review serves as a foundation for the subsequent analyses and discussions presented in this thesis, offering readers a nuanced understanding of the model's significance and relevance within the broader scientific landscape.

In Chapter 3, we consider a mathematical model describing the aggregation of multiple kinds of cells when the response to a chemical signal undergoes a rotation. We were able to find sharp conditions on the initial data for deciding if we have either global existence or blow up in finite time of solutions.

In addition, we were able to apply the proposed technique to solve an open question regarding the possibility of having blow up for a system describing the dynamics of two-species Brownian vortices with different signs, cf. [82, p. 174]. In Chapter 4, motivated by the dynamics of circulating tumor cells, we constructed a mathematical model considering the tensorial attraction of two types of cells and two types of chemicals. We obtained a sharp result of global existence and blow-up for this system.

In Chapter 5, we extended the theory of blow-up of solutions for the Keller-Segel parabolic-elliptic system when there is a tensorial flux induced by chemotactic signals of the form $\chi A\nabla v$, where $A \in M_2(\mathbb{R})$ represents an arbitrary matrix with constant components satisfying $Tr(A), \det(A) > 0$. We obtained this result by designing a new technique to prove blow-up using the decomposition of the matrix A into its polar form and analyzing the evolution of a generalized second moment associated with the system. In Chapter 6, we provided evidence that finite-time blowup solutions are indeed possible in dimensions $n \geq 3$, when utilizing a tensorial flux expressed in the form of $A\nabla v$, where A denotes a matrix with constant components and satisfies quite general conditions.

In Chapter 7, we explored the impact of topographical obstacles on chemotaxis. Our approach entails modifying the Keller-Segel model to incorporate a spatially dependent coefficient of chemotaxis. Through our analysis, we illustrate the critical role of this coefficient in preventing blow-up phenomena in cell concentration.

Chapter 2

Literature review

Chemotaxis is a fundamental process that facilitates the aggregation of species. It is characterized by the movement of organisms toward a concentration gradient of chemicals. Keller and Segel proposed a well-known model for chemotaxis (e.g. [52]). A simplified but still biologically meaningful version of this model is given by the system

$$\begin{aligned} u_t &= \Delta u - \chi \nabla \cdot (u \nabla v), \\ \varepsilon v_t &= \Delta v - v + u, \end{aligned} \tag{2.1}$$

where $u(x,t)$ denotes the density of cells and $v(x,t)$ the chemical concentration at a given point x and time t. About the Keller-Segel model (2.1), Nanjundiah [67] and Childress-Percus [26] introduced the following conjecture:

- In an one-dimensional domain setting, the solution exists globally in time.
- In a two-dimensional domain setting, there exists a critical mass M_c such that if an initial data u_0 satisfies $\int_\Omega u_0 dx < M_c$ then the corresponding solution exists globally in time, and for any $M > M_c$ there are initial data u_0, v_0 such that $\int_\Omega u_0 dx = M$ and the corresponding solution blows up in finite time.
- In a higher-dimensional domain setting, blow-up can occur even though $\int_\Omega u_0 dx$ is small.

The verification of the Childress-Percus conjecture has brought the attention of the research community and thousands of papers analyzing the qualitative behavior of the Keller-Segel model on bounded domain or the whole space case, for radial and nonradial solutions, have been published (e.g. [15, 29, 32, 47, 63, 66]).

Focusing on the two-dimensional parabolic-elliptic Keller-Segel model

$$\begin{aligned} u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) & x \in \mathbb{R}^2, t > 0, \\ -\Delta v &= u, v(x,t) = -\tfrac{1}{2\pi} \int \ln |x-y| \, u(y,t) dy & x \in \mathbb{R}^2, t > 0. \end{aligned} \tag{2.2}$$

Under the assumptions of the initial data

$$u_0 \in L^1_+ \left(\mathbb{R}^2, \left(1 + |x|^2\right) dx \right), u_0 \ln u_0 \in L^1 \left(\mathbb{R}^2, dx \right), \tag{2.3}$$

Blanchet, Dolbeault and Perthame [15] proved that the condition on the initial data $\int_{\mathbb{R}^2} u_0 dx < 8\pi/\chi$ implies the existence of global solutions meanwhile when $\int_{\mathbb{R}^2} u_0 > 8\pi/\chi$ the blow-up of solutions in finite time is possible. The key idea is based on the bounds of the *free energy functional* defined by

$$E(t) := \int_{\mathbb{R}^2} u \ln u dx - \frac{\chi}{2} \int_{\mathbb{R}^2} uv dx.$$

The free energy functional is a well known tool that was introduced for chemotactic models by Nagai, Senba and Yoshida in [66], by Biler in [8], and by Gajewski and Zacharias in [43]. The first term in E is called the *entropy* and second is called a *potential energy.* They gave a priori estimates for the entropy to prove global existence (*"entropy method"*) by using the *dissipation inequality*

$$E(t) + \int_0^t \int_{\mathbb{R}^2} u \left|\nabla \ln u - \chi \nabla v\right|^2 dxds \leq E(0), \tag{2.4}$$

and the next two-dimensional version of the logarithmic Hardy-Littlewood-Sobolev inequality [18], for non-negative $f \in L^1(\mathbb{R}^2)$ such that $f \ln f$ and $f \ln(1 + |x|^2)$ belong to $L^1(\mathbb{R}^2)$. If $\int_{\mathbb{R}^2} f dx = M > 0$, then

$$\begin{aligned} \frac{M}{2} \int_{\mathbb{R}^2} f \ln f \, dx + \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \ln|x-y| \, dy \, dx \\ \geq C(M) := \frac{M^2}{2}(1 + \ln \pi + \ln M). \end{aligned} \tag{2.5}$$

They analyzed the quantity $\int_{\mathbb{R}^2} u(x,t) \left|x\right|^2 dx$ for proving blow-up (*"second moment technique"*) by using the identity

$$\frac{d}{dt} \int_{\mathbb{R}^2} u(x,t) \left|x\right|^2 dx = 4M \left(1 - \frac{\chi M}{8\pi}\right).$$

This approach operates on frameworks of free-energy solutions, which are non-negative weak solutions such that u satisfies

$$\left(1 + |x|^2 + |\ln u|\right) u \in L^\infty_{loc}\left(\mathbb{R}^+, L^1\left(\mathbb{R}^2\right)\right),$$

and the *dissipation inequality* (2.4). Following [77], we say that $u \in L^\infty_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$ is a weak solution to (2.2) with initial data u_0 if for all test functions $\varphi \in C_0^\infty(\mathbb{R}^2)$

$$\begin{aligned} &\int_{\mathbb{R}^2} \varphi u(x,t) dx - \int_{\mathbb{R}^2} \varphi u_0(x) dx \\ &= \int_0^t \int_{\mathbb{R}^2} \Delta\varphi(x) u(x,t) dx d\tau \\ &- \frac{\chi}{4\pi} \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} [\nabla\varphi(x) - \nabla\varphi(y)] \cdot \frac{x-y}{|x-y|^2} u(x,t) u(y,t) dx dy d\tau. \end{aligned}$$

Moreover, In [14] was proved that the critical case $\int_{\mathbb{R}^2} u_0 dx = 8\pi/\chi$ also implies the existence of global free-energy solutions with initial data satisfying (2.3).

It is worth mentioning that many authors have endeavored to avoid the assumptions (2.3) necessary to apply the free energy framework (e.g., [64, 65, 85]). A notable achievement in this sense was reported in [85], being proved that for all non-negative initial data in $L^1(\mathbb{R}^2)$, the global existence holds if and only if the total mass $M \leq 8\pi$. The proof of this result relies on monotonicity formulas derived from nonnegative mild solutions. We say that a function u on $[0,T)\times\mathbb{R}^2$ is a mild solution of (2.2) on $[0,T)$ with initial data $u_0 \in L^1(\mathbb{R}^2)$ if

$$u \in C\left([0,T);L^1\left(\mathbb{R}^2\right)\right) \cap C\left((0,T);L^{4/3}\left(\mathbb{R}^2\right)\right), \sup_{0<t<T} t^{1/4}\left\|u_i(t)\right\|_{L^{4/3}(\mathbb{R}^2)} < \infty,$$

and $u(t)$ satisfies the following Duhamel integral equation for all $t \in (0,T)$.

$$u(t) = e^{t\Delta}u_0 - \chi \int_0^t e^{(t-s)\Delta}\nabla\cdot\left(u(s)\nabla v(s)\right) ds,$$

where

$$\nabla v(x,s) := -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} u(y,s)dy,$$

and $e^{t\Delta}$ is the heat semigroup defined by

$$e^{t\Delta}f := G_t * f, \quad G_t := \frac{1}{4\pi t}e^{-|x|^2/4t}.$$

On the other hand, a number of studies and modeling approaches indicate that external influences may severely affect cells' responses to chemicals in the environment. For example, when swimming bacteria like E. coli or Salmonella swim near a surface, they may undergo a rotational force and form spiral patterns (e.g. [42]). Consequently, it has been found that rotational flow components may also participate in chemotactic migration despite being oriented along the gradient of the chemical substance. In this sense, the next mathematical model was proposed in [93] to describe the dynamics of E. coli when swimming near to a surface

$$\begin{array}{ll} \partial_t u = \Delta u - \nabla\cdot\left(u(\chi_1\nabla v + \chi_2\nabla^{\perp}v)\right), & x \in R^2, t > 0, \\ -\Delta v = u, & x \in R^2, t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in R^2. \end{array}$$

Here u denotes cell density, v denotes chemical concentration, χ_1, χ_2 are positive constants.

We can rewrite the first equation in the previous mathematical model in the form

$$\partial_t u = \Delta u - \nabla\cdot\left(u\begin{pmatrix}\chi_1 & -\chi_2 \\ \chi_2 & \chi_1\end{pmatrix}\nabla v\right),$$

or equivalently

$$\partial_t u = \Delta u - \chi\nabla\cdot\left(uA\nabla v\right),$$

with $\chi = \sqrt{\chi_1^2+\chi_2^2} > 0$ and

$$A := \begin{pmatrix}\chi_1/\chi & -\chi_2/\chi \\ \chi_2/\chi & \chi_1/\chi\end{pmatrix}$$

The matrix A is a rotation matrix and therefore it can be written in the form

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

with $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is fixed.

Therefore the system becomes

$$\begin{array}{ll} \partial_t u = \Delta u - \chi \nabla \cdot (uA\nabla v), & x \in \mathbb{R}^2, t > 0, \\ -\Delta v = u, & x \in \mathbb{R}^2, t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^2. \end{array}$$

Considerable mathematical challenges arise from the rotational dynamics introduced by this model. One key reason is that rotational fluxes in chemotaxis systems complicate the construction of an energy framework suitable for the qualitative analysis of these models. Analyzing the possibility of having solutions blowing-up in finite time has proven particularly challenging. To the best of our knowledge, the first result demonstrating that a rotated chemotactic response can delay or even prevent blow-up was published in [38]. It was proved that if $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and the initial data u_0 satisfies $0 \leq \int_{\mathbb{R}^2} u_0(x)dx < \frac{8\pi}{\chi \cos\alpha}$, then the solution exists globally in time, meanwhile the condition $\int_{\mathbb{R}^2} u_0(x)dx > \frac{8\pi}{x \cos\alpha}$ implies the possibility of having blow-up. Meanwhile it was shown that in the case $\alpha \in \left(-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]$ the solution exists globally in time independently on the value of $\int_{\mathbb{R}^2} u_0(x)dx$. It was also shown that for any angle of rotation there is a dissipative energy structure.

Many papers have been published in the scientific literature describing enough conditions for the global existence of solutions for this kind of models with tensorial chemotaxis of the form $A\nabla v$, where $A := A(x,u,v)$ represents an arbitrary matrix, see for instance [58, 84, 86, 87] and the references therein. However, the possibility of having solutions blowing-up in finite time remains unclear when the chemoattractant is being produced by the cells itself. The only case well-understood in this sense is for the case A being an rotational matrix [38].

It is important to note that in nature, multiple types of cells often interact. Essentially, a mathematical model representative of this situation, together with the qualitative analysis, will be more complex, but at the end, it will be biologically more relevant. One main example is the next two species Keller-Segel model describing the interaction between two kinds of cells with densities u_1 and u_2 and one chemoattractant v presented in [28]

$$\begin{array}{ll} \partial_t u_1 = \mu \Delta u_1 - \chi_1 \nabla \cdot (u_1 \nabla v) & x \in \mathbb{R}^2, t > 0, \\ \partial_t u_2 = \Delta u_2 - \chi_2 \nabla \cdot (u_2 \nabla v) & x \in \mathbb{R}^2, t > 0, \\ v(x,t) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x-y| \left(u_1(y,t) + u_2(y,t)\right) dy & x \in \mathbb{R}^2, t > 0, \\ u_1(x,0) = u_{10} \geq 0, u_2(x,0) = u_{20} \geq 0, & x \in \mathbb{R}^2, \end{array} \tag{2.6}$$

where μ, χ_1, χ_2 represent positive constants. About the multi-species Keller-Segel model in a two-dimensional domain, Wolansky [89] formulated the following natural question: What is the analogue for the critical mass obtained for the self-attracting single species system? A partial answer was given for the model (2.6) in 2011 (see [28]) showing that any of the inequalities

$$\theta_1 > \frac{8\pi\mu}{\chi_1}, \text{ or } \theta_2 > \frac{8\pi}{\chi_2}, \tag{2.7}$$

or

$$\frac{4\pi\mu}{\chi_1}\theta_1 + \frac{4\pi}{\chi_2}\theta_2 - \frac{1}{2}(\theta_1+\theta_2)^2 < 0, \tag{2.8}$$

implies the possibility of having blow-up. Here $\theta_1 = \|u_{10}\|_{L^1(\mathbb{R}^2)}, \theta_2 = \|u_{20}\|_{L^1(\mathbb{R}^2)}$ denote the total initial mass of each species. For condition (2.8), the key idea of the proof is based on the monotonicity of the *second moment for the whole population* defined

$$m(t) := \frac{\pi}{\chi_1}\int_{\mathbb{R}^2} u_1(x,t)\,|x|^2\,dx + \frac{\pi}{\chi_2}\int_{\mathbb{R}^2} u_2(x,t)\,|x|^2\,dx, \tag{2.9}$$

which is strictly decreasing in the region defined by equation (2.8) and increasing otherwise, due to

$$\frac{d}{dt}m(t) = \frac{4\pi\mu}{\chi_1}\theta_1 + \frac{4\pi}{\chi_2}\theta_2 - \frac{1}{2}(\theta_1+\theta_2)^2.$$

This idea is a generalization of the usual technique of the second moments (cf. [15]) for proving blowup for single species to the multi-species case. However, the authors proved that the second moment (2.9) can be increasing and the solutions of system (2.6) can still blow-up by considering the radial case. In fact, for initial radial conditions u_{10}, u_{20}, the second moments $m_1(t), m_2(t)$ with respect to the origin for each variable defined by

$$m_i(t) := \int_{\mathbb{R}^2} u_i(x,t)\,|x|^2\,dx, i = 1,2,$$

satisfy

$$\frac{d}{dt}m_1(t) \le 4\theta_1\left(1 - \frac{\chi_1\theta_1}{8\pi\mu}\right) \text{ and } \frac{d}{dt}m_2(t) \le 4\theta_2\left(1 - \frac{\chi_2\theta_2}{8\pi}\right).$$

The authors also proved that the system (2.6) has a dissipative energy structure. i.e., they defined the following free-energy functional

$$\begin{aligned} E(t) := &\frac{\mu}{\chi_1}\int_{\mathbb{R}^2} u_1\log u_1 dx + \frac{1}{\chi_2}\int_{\mathbb{R}^2} u_2\log u_2 dx \\ &- \frac{1}{2}\int_{\mathbb{R}^2} u_1 v dx - \frac{1}{2}\int_{\mathbb{R}^2} u_2 v dx, \end{aligned}$$

and proved that it satisfies

$$\begin{aligned} \frac{d}{dt}E(t) = &-\frac{1}{\chi_1}\int_{\mathbb{R}^2} u_1\,|\mu\nabla\log u_1 - \nabla\chi_1 v|^2\,dx \\ &- \frac{1}{\chi_2}\int_{\mathbb{R}^2} u_2\,|\nabla\log u_2 - \nabla\chi_2 v|^2\,dx \\ &\le 0. \end{aligned}$$

Based on the monotonicity of the free-energy functional $E(t)$ and the logarithmic HLS inequality (2.5), it was also proved that the inequalities

$$\theta_1 + \theta_2 < \frac{8\pi}{\chi_2},\ \ \mu \geq 1,$$
$$\theta_1 + \theta_2 < \frac{8\pi}{\chi_2}\mu,\ \ \mu < 1,$$

guarantee global existence. In [36], the authors improved the results of global existence from [28] by the two-dimensional version of the logarithmic Hardy-Littlewood-Sobolev inequality for systems (See [78, Theorem 4.]) to deduce that the inequalities

$$\theta_1 < \frac{8\pi\mu}{\chi_1}, \theta_2 < \frac{8\pi}{\chi_2},\ \text{and}\ \frac{4\pi\mu}{\chi_1}\theta_1 + \frac{4\pi}{\chi_2}\theta_2 - \frac{1}{2}\left(\theta_1 + \theta_2\right)^2 > 0,$$

guarantee global existence. In consequence, there exists a critical curve in the plane of initial masses $\theta_1\theta_2$ delimiting on one side global existence and blow-up on the other side.

Chapter 3

Optimal critical mass for the multi-species Keller-Segel model with rotational flux terms

Abstract

This chapter demonstrates how a single Keller-Segel model with rotational flux terms can address questions regarding the global existence and blow-up of solutions for several other Keller-Segel-type models arising in mathematical biology and physics. In the case of aggregating two species on a two-dimensional domain, a threshold curve is identified in the plane of masses that allows determination of whether the solution of the system blows up or remains global in time. Additionally, this research provides a novel blow-up result for a mathematical model recently introduced in the literature. The research discussed in this chapter has been accepted for publication in the journal *Differential and Integral Equations* (Volume 37, Issues 11-12, November/December 2024) under the title: *Optimal Critical Mass for the Multi-species Keller-Segel Model with Rotational Flux Terms.*

A number of studies and modeling approaches indicate that external influences may severely affect cells' responses to chemicals in the environment. Consequently, it has been found that rotational flow components may also participate in chemotactic migration despite being oriented along the gradient of the chemical substance. A detailed discussion of this subject can be found in [92] and [93]. The two-dimensional study of rotational flows is particularly important in mathematical modeling when considering thin layers or rapidly rotating fluids, where the Coriolis force strongly disfavors displacement along the axis of rotation. Geophysical flows and the effects of the Earth's rotation are two interesting examples of how a two-dimensional approximation can be accurate and useful, cf. [24]. Because of this rotational dynamics significant mathematical difficulties arise in the analysis of cells' aggregation surrounded by these kind of fluids. The reason for this is that rotational fluxes in chemotaxis systems make it hard to find an energy functional that could be used in the qualitative analysis of these models. Analyzing a possible blow-up, in particular, has been challenging. To the best of our knowledge, the first result showing that rotated chemotactic response can delay or even avoid the blow-up was published in [38]. It was also shown that for any angle of rotation there is a dissipative energy structure. The purpose of this chapter is to extend the analysis in [38] from the one-species to the multi-species case. In this context, we take the next Keller-Segel model with rotational flux terms

$$\begin{array}{ll} \partial_t u_1 = \mu_1 \triangle u_1 - \chi_1 \nabla \cdot (u_1 A_1 \nabla v) & x \in \mathbb{R}^2, \quad t > 0, \\ \partial_t u_2 = \mu_2 \triangle u_2 - \chi_2 \nabla \cdot (u_2 A_2 \nabla v) & x \in \mathbb{R}^2, \quad t > 0, \\ -\triangle v = a_1 u_1 + a_2 u_2, & x \in \mathbb{R}^2, \quad t > 0, \\ u_1(x,0) = u_{10} \geq 0, u_2(x,0) = u_{20} \geq 0 & x \in \mathbb{R}^2. \end{array} \tag{3.1}$$

where

$$A_1 = \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ \sin\alpha_1 & \cos\alpha_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \cos\alpha_2 & -\sin\alpha_2 \\ \sin\alpha_2 & \cos\alpha_2 \end{pmatrix}, \tag{3.2}$$

the parameters $\mu_1, \mu_2, \chi_1, \chi_2$ represent positive constants, $a_1, a_2 \in \mathbb{R}$ are arbitrary constants non-vanishing simultaneously, and $\alpha_1, \alpha_2 \in (-\pi, \pi]$ are fixed. In the context of mathematical biology, u_1 and u_2 can be interpreted as the density variables for two different species meanwhile v can be interpreted as the concentration of a chemoattractant. In system (3.1), the equations for u_1 and u_2 indicate that the motion of both species is driven by self-diffusion and the gradient of concentration of the chemical, while the equation for the concentration of the chemical describes that it is either produced or consumed by the species, depending on the sign of the coefficients a_1 and a_2, and it is diffusing into the environment.

Taking into account that the elliptic equation for v does not have unique solutions, we will work simply with the fundamental solution

$$v = -\frac{1}{2\pi} \ln|\cdot| * (a_1 u_1 + a_2 u_2). \tag{3.3}$$

We also assume that we have nonnegative initial data u_{10}, u_{20} satisfying

$$\theta_1 := \int_{\mathbb{R}^2} u_{10}(x)dx > 0 \quad \text{and} \quad \theta_2 := \int_{\mathbb{R}^2} u_{20}(x)dx > 0. \tag{3.4}$$

A remarkable property of this mathematical model is that it rescues several classical mathematical models in biology and physics when we change the value of the parameters arising in this model. For instance, the case $a_2 = 0$ gives

$$\begin{aligned} \partial_t u_1 &= \mu_1 \triangle u_1 - \chi_1 \nabla \cdot (u_1 A_1 \nabla v), \\ -\triangle v &= a_1 u_1, \end{aligned}$$

which corresponds to the one species Keller-Segel model with rotational flux terms proposed in [38]. In particular, the case $\alpha_1 = a_2 = 0$ is very interesting since it rescues the classical parabolic-elliptic Keller-Segel model. As shown in [15] a blow-up of the solutions is possible in this case. On the other hand, taking $\alpha_1 = \pi$ and $a_2 = 0$, we obtain the Keller-Segel model with negative chemotaxis, which never blows-up. Surprisingly, this model also includes as a particular case the vorticity equation in fluid dynamics which arises for the values of the parameters $\alpha_1 = -\pi/2$, $\mu_1 = \chi_1 = 1$ and $a_2 = 0$.

It must be noticed that in nature quite often not only one but several kinds of cells interact. The analysis of a mathematical model describing this interaction is typically more complex, but closer to reality in the context of biology. One main example is the next two species Keller-Segel model describing the interaction between two kinds of cells with densities u_1 and u_2 and one chemoattractant v

$$\begin{aligned} \partial_t u_1 &= \mu_1 \triangle u_1 - \chi_1 \nabla \cdot (u_1 \nabla v), \\ \partial_t u_2 &= \mu_2 \triangle u_2 - \chi_2 \nabla \cdot (u_2 \nabla v), \\ -\triangle v &= u_1 + u_2. \end{aligned}$$

The qualitative analysis that outlines the conditions for having global existence of solutions or blow-up for this model can be found in [36]. Notice that this model arises, as a particular case, from our proposed mathematical model (3.1) by taking $\alpha_1 = \alpha_2 = 0$ and $a_1 = a_2 = \mu_2 = 1$. A relevant highlight of system (3.1) is that it not only rescues several well-known mathematical models as particular cases, but it also includes new models whose theory of global existence of solutions has not yet been developed. For instance, consider the system

$$\begin{aligned} \partial_t u_1 &= \mu_1 \triangle u_1 - \overline{\chi}_1 \nabla \cdot (u_1 \nabla v) - \nabla \cdot (u_2 \nabla^{\perp} v), \\ \partial_t u_2 &= \mu_2 \triangle u_2 - \overline{\chi}_2 \nabla \cdot (u_2 \nabla v) - \nabla \cdot (u_2 \nabla^{\perp} v), \\ -\triangle v &= u_1 + u_2, \end{aligned}$$

with positive parameters $\mu_1, \mu_2, \overline{\chi}_1$, and $\overline{\chi}_2$. It describes the dynamics of two types of bacteria producing chemoattractant and swimming near a surface, where they may be subject to a net rotational force and form spirals, cf. [5, 48]. We realize that this system has the same structure as the proposed model (3.1) when taking

$$A_1 = \begin{pmatrix} \overline{\chi}_1/\sqrt{1+\overline{\chi}_1^2} & -1/\sqrt{1+\overline{\chi}_1^2} \\ 1/\sqrt{1+\overline{\chi}_1^2} & \overline{\chi}_1/\sqrt{1+\overline{\chi}_1^2} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \overline{\chi}_2/\sqrt{1+\chi_2^2} & -1/\sqrt{1+\overline{\chi}_2^2} \\ 1/\sqrt{1+\overline{\chi}_2^2} & \overline{\chi}_2/\sqrt{1+\overline{\chi}_2^2} \end{pmatrix},$$

and coefficients $\chi_1 = \sqrt{1+\overline{\chi}_1^2}$ and $\chi_2 = \sqrt{1+\overline{\chi}_2^2}$. As far as we know, neither the global existence nor the blow-up of solutions has been reported. Another interesting example is the mathematical model

$$\begin{aligned} \partial_t u_1 &= \mu_1 \triangle u_1 - \chi_1 \nabla \cdot (u_1 \nabla v) - \nabla \cdot (u_2 \nabla^{\perp} v), \\ \partial_t u_2 &= \mu_2 \triangle u_2 + \chi_2 \nabla \cdot (u_2 \nabla v) - \nabla \cdot (u_2 \nabla^{\perp} v), \\ -\triangle v &= u_1 - u_2. \end{aligned} \tag{3.5}$$

This system was proposed in [23, Eqs. 120-122] to describe the dynamics of two-species Brownian vortices with different signs. It is also worth mentioning that this model constitutes an extension of the Debye-Hueckel model of electrolytes (cf. [10]), where the like charges repel each other. The question of blow-up for the solutions of system (3.5) was proposed in the reference [82,

p. 174] and it remains open up to the best of our knowledge. In summary, the main goal of this chapter is not only to provide new results on global existence and blow-up for several mathematical models, but also to propose a mathematical approach that unifies the theory of global existence of solutions for several Keller-Segel type models and the vorticity equation.

From a mathematical point of view, a main feature in the analysis of system (3.1) is that traditional approaches to constructing energy functional (e.g. [15, 36, 28, 64]) become challenging. A main reason for this is the lack of symmetry caused by tensorial chemoattraction making it difficult to deal with entropy functionals $\int_{\mathbb{R}^2} u_i \ln u_i dx, i = 1,2$. Following a technique introduced in [46], we showed in this chapter how to modify these entropy functionals by another ones that have lower bounds and still allow finding optimal conditions on the initial data to obtain global solutions. Similarly, due to tensorial chemotaxis, it is difficult to find conditions that guarantee a blow-up of the solutions. Nevertheless, we find conditions on radial initial data that allow us to decide whether the solutions of model (3.1) blow up within finite time.

Introducing our results, let us begin by defining a weak solution for (3.1).

Definition 1 (Weak solution) *Let* $\mu_1, \mu_2, \chi_1, \chi_2$ *be positive constants, meanwhile* α_1, α_2 *are constants restricted to the interval* $(-\pi, \pi]$ *and* a_1, a_2 *arbitrary constants satisfying* $a_1^2 + a_2^2 > 0$. *Let* A_i *with* $i = 1,2$ *be the* 2×2 *matrices defined by (3.2). Given* $T > 0$, *the vector-valued function* (u_1, u_2) *is a* ***weak solution*** *on* $\mathbb{R}^2 \times (0,T)$ *of system (3.1), with initial data satisfying*

$$0 \le u_{i0} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2),\ u_{i0} \ln u_{i0} \in L^1(\mathbb{R}^2),\ u_{i0} \ln(1+|x|^2) \in L^1(\mathbb{R}^2), \tag{3.6}$$

for $i = 1,2$, *if*

i) $u_i \in C\left([0,T); L^1\left(\mathbb{R}^2\right)\right) \cap L^{4/3}((0,T) \times L^{4/3}(\mathbb{R}^2))$, *and*

ii) (u_1, u_2) *verify (3.1) in the weak sense, that is to say*

$$\begin{aligned}&\int_{\mathbb{R}^2} \varphi u_i(x,t) dx - \int_{\mathbb{R}^2} \varphi u_{i0}(x) dx \\ &= \mu_i \int_0^t \int_{\mathbb{R}^2} u_i \Delta\varphi dx d\tau + \chi_i \int_0^t \int_{\mathbb{R}^2} \nabla\varphi \cdot (u_i A_i \nabla v)\, dx d\tau,\end{aligned}$$

for $i = 1,2$, *for any* $\varphi \in C_0^\infty(\mathbb{R}^2)$, $0 < t < T$, *and for* v *being defined by (3.3).*

Our main result of global existence now read as follows.

Theorem 2 (Global existence) *Assume that* a_1, a_2 *are non-negative constants. Let us denote by* θ_i *with* $i = 1,2$ *the total initial masses define by (3.4).*

1. *If* $\alpha_1, \alpha_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, *i.e., both species move toward the gradient of chemical concentration and* θ_i *with* $i = 1,2$ *satisfy the inequalities*

$$\begin{aligned}&\theta_1 < \frac{8\pi\mu_1}{\chi_1 a_1 \cos\alpha_1},\ \theta_2 < \frac{8\pi\mu_2}{\chi_2 a_2 \cos\alpha_2}, \\ &\text{and } \frac{8\pi\mu_1 a_1}{\chi_1 \cos\alpha_1}\theta_1 + \frac{8\pi\mu_2 a_2}{\chi_2 \cos\alpha_2}\theta_2 - (a_1\theta_1 + a_2\theta_2)^2 > 0,\end{aligned} \tag{3.7}$$

then system (3.1) has a global weak solution.

2. *If* $\alpha_1, \alpha_2 \in \left(-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]$*, i.e., both species move away of the gradient of chemical concentration, then for any initial masses* θ_i*,* $i = 1, 2$*, the system (3.1) has a global weak solution*

On the other hand, assuming $\cos\alpha_1 \cos\alpha_2 > 0$ *and* $u_{10}, u_{20} \in L^1(\mathbb{R}^2, (1 + |x|^2)dx)$*, and denoting by sgn the sign function, we have that the free-energy functional defined by*

$$\begin{aligned} E(t) := \frac{\mu_1 a_1}{\chi_1 |\cos\alpha_1|} \int_{\mathbb{R}^2} u_1 \ln u_1 dx + \frac{\mu_2 a_2}{\chi_2 |\cos\alpha_2|} \int_{\mathbb{R}^2} u_2 \ln u_2 dx \\ - \frac{sgn(\cos\alpha_1) a_1}{2} \int_{\mathbb{R}^2} u_1 v dx - \frac{sgn(\cos\alpha_2) a_2}{2} \int_{\mathbb{R}^2} u_2 v dx, \end{aligned} \tag{3.8}$$

satisfies the following dissipation inequality

$$\begin{aligned} E(t) + \frac{a_1}{\chi_1 |\cos\alpha_1|} \int_0^t \int_{\mathbb{R}^2} u_1 \left|\nabla(\mu_1 \ln u_1 - \chi_1 \cos\alpha_1 v)\right|^2 dx \\ + \frac{a_2}{\chi_2 |\cos\alpha_2|} \int_0^t \int_{\mathbb{R}^2} u_2 \left|\nabla(\mu_2 \ln u_2 - \chi_2 \cos\alpha_2 v)\right|^2 dx \le E(0). \end{aligned} \tag{3.9}$$

Remark 3 *Notice that the inequality*

$$\frac{8\pi\mu_1 a_1}{\chi_1 \cos\alpha_1}\theta_1 + \frac{8\pi\mu_2 a_2}{\chi_2 \cos\alpha_2}\theta_2 - (a_1\theta_1 + a_2\theta_2)^2 > 0, \tag{3.10}$$

corresponds to the interior of a rotated parabola in the plane $\theta_1\theta_2$*. Choosing the parameters appropriately, conditions* $\theta_1 < \frac{8\pi\mu_1}{\chi_1 a_1 \cos\alpha_1}$ *and* $\theta_2 < \frac{8\pi\mu_2}{\chi_2 a_2 \cos\alpha_2}$ *may be relevant or can be simply ignored. In fact, if the parabola*

$$\frac{8\pi\mu_1 a_1}{\chi_1 \cos\alpha_1}\theta_1 + \frac{8\pi\mu_2 a_2}{\chi_2 \cos\alpha_2}\theta_2 - (a_1\theta_1 + a_2\theta_2)^2 = 0,$$

does not intersect either of the lines $\theta_1 = \frac{8\pi\mu_1}{\chi_1 a_1 \cos\alpha_1}$ *or* $\theta_2 = \frac{8\pi\mu_2}{\chi_2 a_2 \cos\alpha_2}$ *in the first quadrant of the plane* $\theta_1\theta_2$ *(when* $\frac{\mu_1}{2\mu_2} \le \frac{\chi_1 \cos\alpha_1}{\chi_2 \cos\alpha_2} \le \frac{2\mu_1}{\mu_2}$*), then inequality (3.10) is enough to guarantee the global existence result in Theorem 2 item 1.*

Moreover, we observe the following phenomena of blow-up.

Theorem 4 (Blow-up) *Let us denote by* θ_i *with* $i = 1, 2$ *the total initial masses define by (3.4). Consider a weak solution* (u_1, u_2) *of system (3.1) and let* $[0, T_{\max})$ *be the corresponding maximal interval of existence. Assume that the initial data* u_{10}, u_{20} *satisfy (3.6) and radially symmetric and* $u_{10}, u_{20} \in L^1(\mathbb{R}^2, (1 + |x|^2)dx)$*. Then, we have*

1. *if* $a_1 \cos\alpha_1, a_2 \cos\alpha_2 > 0$ *and the initial masses* $\theta_i, i = 1, 2,$ *satisfy the inequality*

$$\frac{8\pi\mu_1 a_1}{\chi_1 \cos\alpha_1}\theta_1 + \frac{8\pi\mu_2 a_2}{\chi_2 \cos\alpha_2}\theta_2 - (a_1\theta_1 + a_2\theta_2)^2 < 0, \tag{3.11}$$

then $T_{\max} < \infty$*;*

2. *if* $a_1 a_2 \geq 0$, $a_i \cos\alpha_i > 0$ *for at least one index* $i \in \{1,2\}$ *and the initial mass* θ_i *satisfies the inequality*

$$\theta_i > \frac{8\pi\mu_i}{\chi_i a_i \cos\alpha_i}, \tag{3.12}$$

then $T_{\max} < \infty$.

Remark 5 *Notice that this result includes the model (3.5), which describes the dynamics of two-species Brownian vortices with different signs, since in this case* $a_1 = 1, a_2 = -1$ *and*

$$A_1 = \begin{pmatrix} \chi_1/\sqrt{1+\chi_1^2} & -1/\sqrt{1+\chi_1^2} \\ 1/\sqrt{1+\chi_1^2} & \chi_1/\sqrt{1+\chi_1^2} \end{pmatrix},$$
$$A_2 = \begin{pmatrix} -\chi_2/\sqrt{1+\chi_2^2} & -1/\sqrt{1+\chi_2^2} \\ 1/\sqrt{1+\chi_2^2} & -\chi_2/\sqrt{1+\chi_2^2} \end{pmatrix}.$$

To the best of our knowledge, this is the first result of blow-up reported for this system.

Corollary 6 *Assume that* a_1, a_2 *are non-negative constants and* $\alpha_1, \alpha_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$*, i.e., both species move toward the gradient of chemical concentration. Consider a weak solution* (u_1, u_2) *of system (3.1) and let* $[0, T_{\max})$ *be the corresponding maximal interval of existence. Assume that the initial data are non-negative, radially symmetric and* $u_{10}, u_{20} \in L^1(\mathbb{R}^2, (1+|x|^2)dx)$*. If* θ_1 *and* θ_2 *satisfy any of the inequalities*

$$\frac{8\pi\mu_1 a_1}{\chi_1\cos\alpha_1}\theta_1 + \frac{8\pi\mu_2 a_2}{\chi_2\cos\alpha_2}\theta_2 - (a_1\theta_1 + a_2\theta_2)^2 < 0, \tag{3.13}$$

$$\text{or} \quad \theta_1 > \frac{8\pi\mu_1}{\chi_1 a_1\cos\alpha_1}, \quad \text{or} \quad \theta_2 > \frac{8\pi\mu_2}{\chi_2 a_2\cos\alpha_2}, \tag{3.14}$$

then, $T_{\max} < \infty$.

Remark 7 *Note that if* $\alpha_1 = \alpha_2 = 0, \mu_1 = \mu$, $\mu_2 = 1$ *and* $a_1 = a_2 = 1$*, we rescue the two-species Keller-Segel model with one chemical in* $\mathbb{R}^2$ *proposed in [28]*

$$\begin{aligned} &\partial_t u_1 = \mu\triangle u_1 - \chi_1\nabla\cdot(u_1\nabla v) && x\in\mathbb{R}^2, t>0,\\ &\partial_t u_2 = \triangle u_2 - \chi_2\nabla\cdot(u_2\nabla v) && x\in\mathbb{R}^2, t>0,\\ &-\triangle v = u_1 + u_2, && x\in\mathbb{R}^2, t>0. \end{aligned}$$

Corollary 6 guarantees that we can always construct initial data with masses θ_1 *and* θ_2 *such that if they satisfy any of the inequalities*

$$\frac{8\pi\mu}{\chi_1}\theta_1 + \frac{8\pi}{\chi_2}\theta_2 - (\theta_1+\theta_2)^2 < 0, \ \text{or } \theta_1 > \frac{8\pi\mu}{\chi_1}, \ \text{or } \theta_2 > \frac{8\pi}{\chi_2},$$

then $T_{\max} < \infty$*, which coincides with the sharp result given in [36, 28].*

Remark 8 *In the case $a_2 = 0$, $a_1 > 0$, we rescue the one species Keller-Segel model with rotational flux terms*

$$\begin{aligned} \partial_t u_1 &= \mu_1 \triangle u_1 - \chi_1 \nabla \cdot (u_1 A_1 \nabla v), \\ -\triangle v &= a_1 u_1, \end{aligned}$$

proposed in [38]. Corollary 6 and Theorem 2 guarantee that if $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and the initial mass θ_1 satisfies $\theta_1 < \frac{8\pi}{\chi \cos\alpha}$, then the corresponding solution exists globally in time, meanwhile the condition $\theta_1 > \frac{8\pi}{x \cos\alpha}$ implies the possibility of having blow-up. On the other hand, Theorem 2 also shows that in the case $\alpha \in \left(-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]$ the solution exists globally in time independently on the value of θ_1. These results coincide with the sharp result given in [38].

3.1 Local existence, uniqueness, regularity, mass conservation and positivity

In light of [44], [51] and [64], we establish the local existence, uniqueness, regularity, mass conservation and positivity of mild solutions to (3.1). This approach piques interest as it solely considers the initial data in $L^1(\mathbb{R}^2)$.

Let us begin by defining a mild solution for (3.1).

Definition 9 (Mild Solution) *Given $u_{10}, u_{20} \in L^1(\mathbb{R}^2)$, we define (u_1, u_2) to be a mild solution of (3.1) on $[0, T)$ with initial data (u_{10}, u_{20}) if*

i) $u_i \in C\left([0,T); L^1\left(\mathbb{R}^2\right)\right) \cap C\left((0,T); L^{4/3}\left(\mathbb{R}^2\right)\right)$ *for $i = 1, 2$,*

ii) $\sup_{0<t<T} t^{1/4} \left\| u_i(t) \right\|_{L^{4/3}(\mathbb{R}^2)} < \infty$ *for $i = 1, 2$,*

iii) (u_1, u_2) *satisfies the following Duhamel integral equations for all $t \in (0, T)$.*

$$u_i(t) = e^{\mu_i t \Delta} u_{i0} - \chi_i \int_0^t \nabla \cdot e^{\mu_i (t-s)\Delta} \left(u_i(s) A_i \nabla v(s) \right) ds, \tag{3.15}$$

for $i = 1, 2$, where $\nabla v := \nabla \mathbf{K} * (a_1 u_1 + a_2 u_2)$, with $\mathbf{K}(x) := -\frac{1}{2\pi} \ln |x|, x \in \mathbb{R}^2 \backslash \{0\}$ and $e^{\mu_i t \Delta}$ is the heat semigroup defined by

$$e^{\mu_i t \Delta} f := G_{t,\mu_i} * f, \quad G_{t,\mu_i} := \frac{1}{4\pi \mu_i t} e^{-|x|^2 / 4\mu_i t}.$$

Moreover, (u_1, u_2) is a global mild solution of (3.1) if (u_1, u_2) is a mild solution of (3.1) on $[0, T)$ for any $0 < T < \infty$.

Let us write (3.15) as

$$(u_1(t), u_2(t)) = \left(e^{\mu_1 t \Delta} u_{10}, e^{\mu_2 t \Delta} u_{20}\right) - B\left((u_1, u_2), (u_1, u_2)\right)(t), 0 < t < T, \tag{3.16}$$

where $B := (\chi_1 B_1, \chi_2 B_2)$ is a bilinear form in which B_1 and B_2 are bilinear forms definded by

$$\begin{aligned} & B_i\left((u_1,u_2),(w_1,w_2)\right)(t) \\ & := \int_0^t \nabla\cdot e^{\mu_i(t-s)\Delta}\left(u_i(s)A_i\left(\nabla\mathbf{K}*(a_1w_1+a_2w_2)\right)(s)\right)ds, \text{ for } i=1,2. \end{aligned} \tag{3.17}$$

For the construction of local-in-time solutions to (3.1), we make now some remarks that will be useful.

First, we recall the following $L^q - L^p$ estimates of heat semigroup $e^{\mu_i t\Delta}$. For any $1 \le q \le p \le \infty$, there holds

$$\left\|e^{\mu_i t\Delta}f\right\|_{L^p(\mathbb{R}^2)} \le (4\pi\mu_i t)^{\frac{1}{p}-\frac{1}{q}}\left\|f\right\|_{L^q(\mathbb{R}^2)}, \tag{3.18}$$

$$\left\|\nabla\cdot e^{\mu_i t\Delta}F\right\|_{L^p(\mathbb{R}^2)} \le Ct^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}\left\|F\right\|_{L^q(\mathbb{R}^2)}, \tag{3.19}$$

where $C = C(p,q,\mu_i)$ is a constant depending only on p,q and μ_i. These inequalities are a consequences of Young's inequality for the convolution (For example, see [45, Subsection 4.1.2. p. 145]).

Secondly, we also recall the following inequality

$$\left\|\frac{1}{|x|}*g\right\|_{L^r(\mathbb{R}^2)} \le C_r\left\|g\right\|_{L^{\frac{2r}{2+r}}(\mathbb{R}^2)}, \text{ for any } r\in(2,+\infty), \tag{3.20}$$

where $C_r = C(r)$ is a constant depending only on r (See [81, Theorem 1 (b), p. 119].

For the case $r=\infty$, we have that for a constant C_q depending only on q,

$$\left\|\frac{1}{|x|}*g\right\|_{L^\infty(\mathbb{R}^2)} \le C_q\left\|g\right\|_{L^1(\mathbb{R}^2)}^{\frac{q-2}{2(q-1)}}\left\|g\right\|_{L^q(\mathbb{R}^2)}^{\frac{q}{2(q-1)}}, \tag{3.21}$$

for all $g\in L^1(\mathbb{R}^2)\cap L^q(\mathbb{R}^2)$, $2<q\le\infty$ (See [64, Lemma 2.5]).

Now, we define the Banach space X_T by

$$X_T := \left\{\begin{array}{c} (u_1(\cdot,t),u_2(\cdot,t)) \in \left(C\left((0,T);L^{4/3}\left(\mathbb{R}^2\right)\right)\right)^2, \\ \sup_{0<t<T} t^{1/4}\left(\left\|u_1(\cdot,t)\right\|_{L^{4/3}(\mathbb{R}^2)}+\left\|u_2(\cdot,t)\right\|_{L^{4/3}(\mathbb{R}^2)}\right)<\infty \end{array}\right\}, \tag{3.22}$$

with norm $\left\|\cdot\right\|_{X_T}$

$$\left\|(u_1,u_2)\right\|_{X_T} := \sup_{0<t<T} t^{1/4}\left(\left\|u_1(\cdot,t)\right\|_{L^{4/3}(\mathbb{R}^2)}+\left\|u_2(\cdot,t)\right\|_{L^{4/3}(\mathbb{R}^2)}\right). \tag{3.23}$$

And we recall well-known simple result on solving quadratic equations like (3.16) via contraction mapping argument (For example, see [64, Lemma 2.3.],[12, Lemma 2.2.] and [57, Theorem 13.2. p. 124]):

Lemma 10 (The Picard contraction principle) *Let X be a Banach space with norm $\|\cdot\|_X$ and $B : X \times X \to X$ a bilinear continuous form satisfying*

$$\|B(u,v)\|_X \leq C_B \|u\|_X \|v\|_X \text{ for all } u, v \in X,$$

where $C_B > 0$ is a constant independent of $u, v \in X$. Then, for every $y \in X$ such that $\|y\|_X < 1/(4C_B)$, the equation $u = y + B(u,u)$ has a solution $u \in X$. Moreover, this solution is such that

$$\|u\|_X \leq \frac{1 - \sqrt{1 - 4C_B \|y\|_X}}{2C_B} \leq 2\|y\|_X$$

and unique in the open ball of radius $1/(2C_B)$. The solution continuously depends on y: If $\|z\|_X \leq \varepsilon < 1/(4C_B)$, $v = z + B(v,v)$ and $\|v\|_X \leq 2\varepsilon$, then

$$\|u - v\|_X \leq (1 - 4C_B\varepsilon)^{-1} \|y - z\|_X .$$

In the following proposition, the local existence in time of mild solutions to (3.1) is proved by applying (10), and also we establish some important properties of the solutions.

Proposition 11 (Local existence) *Given $u_{10}, u_{20} \in L^1(\mathbb{R}^2)$, there exists $T \in (0,\infty)$ such that (3.1) has a unique mild solution (u_1, u_2) on $[0,T)$. Moreover, (u_1, u_2) satisfies the following properties:*

(i) *mass conservation, i.e.,* $\int_{\mathbb{R}^2} u_i dx = \theta_i$, for $i = 1, 2$;

(ii) *integrability, i.e.*, for every $1 \leq p \leq \infty$, there holds $u_i \in C((0,T); L^p(\mathbb{R}^2))$ and $\sup_{0<t<T} \left(t^{1-1/p} \|u_i(t)\|_{L^p(\mathbb{R}^2)}\right) < \infty$,for $i = 1, 2$;

(iii) *decay rates*, i.e., for every $m \in \mathbb{Z}_+, l \in \mathbb{Z}_+^2$ and $1 < p < \infty$, there holds $\partial_t^m \partial_x^l u_i \in C((0,T); L^p(\mathbb{R}^2))$ and $\sup_{0<t<T}(t^{1-1/p+|l|/2+m} \left\|\partial_t^m \partial_x^l u_i\right\|_{L^p(\mathbb{R}^2)}) < \infty$, for $i = 1, 2$;

(iv) *regularity, i.e.,* (u_1, u_2) is a classical solution of (3.1) in $\mathbb{R}^2 \times (0,T)$;

(v) if $u_{10}, u_{20} \in H^1(\mathbb{R}^2)$, then $u_i \in BC([0,T); H^1(\mathbb{R}^2))$, for $i = 1, 2$;

(vi) *positivity, i.e.,* if $u_{10}, u_{20} \geq 0$ and $\theta_1, \theta_2 > 0$, then $u_i > 0$ for all $(x,t) \in \mathbb{R}^2 \times (0,T)$ for $i = 1, 2$;

Proof. *Local existence.* By (3.19), we have

$$\begin{aligned}
&\|B_i((u_1,u_2),(w_1,w_2))\|_{L^{4/3}(\mathbb{R}^2)} \\
&\leq C_1 \int_0^t (t-s)^{-\frac{1}{2}+\frac{3}{4}-1} \left\||u_i(s) A_i (\nabla \mathbf{K} * (a_1 w_1 + a_2 w_2))(s)|\right\|_{L^1(\mathbb{R}^2)} ds \\
&\leq C_1 \int_0^t (t-s)^{-\frac{3}{4}} \|u_i(s)\|_{L^{4/3}(\mathbb{R}^2)} \left\||A_i (\nabla \mathbf{K} * (a_1 w_1 + a_2 w_2))(s)|\right\|_{L^4(\mathbb{R}^2)} ds \\
&= C_1 \int_0^t (t-s)^{-\frac{3}{4}} \|u_i(s)\|_{L^{4/3}(\mathbb{R}^2)} \left\||\nabla \mathbf{K} * (a_1 w_1 + a_2 w_2)(s)|\right\|_{L^4(\mathbb{R}^2)} ds.
\end{aligned}$$

Using (3.20) we obtain

$$\begin{aligned}
&\|B_i((u_1,u_2),(w_1,w_2))\|_{L^{4/3}(\mathbb{R}^2)}\\
&\le \frac{C_1}{2\pi}\int_0^t (t-s)^{-\frac{3}{4}}\|u_i(s)\|_{L^{4/3}(\mathbb{R}^2)}\left\|\frac{1}{|x|}*(a_1w_1+a_2w_2)(s)\right\|_{L^4(\mathbb{R}^2)}ds\\
&\le \frac{C_2}{2\pi}\int_0^t (t-s)^{-\frac{3}{4}}\|u_i(s)\|_{L^{4/3}(\mathbb{R}^2)}\|a_1w_1(s)+a_2w_2(s)\|_{L^{4/3}(\mathbb{R}^2)}ds\\
&\le \frac{C_2}{2\pi}\int_0^t (t-s)^{-\frac{3}{4}}\|u_i(s)\|_{L^{4/3}(\mathbb{R}^2)}\Big(a_1\|w_1(s)\|_{L^{4/3}(\mathbb{R}^2)}+a_2\|w_2(s)\|_{L^{4/3}(\mathbb{R}^2)}\Big)ds\\
&\le \frac{\max\{|a_1|,|a_2|\}C_2}{2\pi}\int_0^t\Big((t-s)^{-\frac{3}{4}}s^{-\frac{1}{4}}s^{-\frac{1}{4}}\Big)ds\,\|(u_1,u_2)\|_{X_T}\|(w_1,w_2)\|_{X_T}\\
&= \frac{\max\{|a_1|,|a_2|\}C_2}{2\pi}\int_0^t\Big((t-s)^{\frac{1}{4}-1}s^{\frac{1}{2}-1}\Big)ds\,\|(u_1,u_2)\|_{X_T}\|(w_1,w_2)\|_{X_T}.
\end{aligned}$$

Using (3.47), we get

$$\begin{aligned}
&\|B_i((u_1,u_2),(w_1,w_2))\|_{L^{4/3}(\mathbb{R}^2)}\\
&\le \frac{C_B}{2\chi_i}t^{-\frac{1}{4}}\|(u_1,u_2)\|_{X_T}\|(w_1,w_2)\|_{X_T},\ \text{for } i=1,2.
\end{aligned}$$

So, we have that

$$\|B((u_1,u_2),(w_1,w_2))\|_{X_T}\le C_B\|(u_1,u_2)\|_{X_T}\|(w_1,w_2)\|_{X_T}, \tag{3.24}$$

where $C_B = C(\chi_1,\chi_2,a_1,a_2,\mu_1,\mu_2)$ is a constant. Moreover, if $u_{10},u_{20}\in L^1(\mathbb{R}^2)$, we claim that

$$t^{1/4}\Big(\big\|e^{\mu_1 t\Delta}u_{10}\big\|_{L^{4/3}(\mathbb{R}^2)}+\big\|e^{\mu_2 t\Delta}u_{20}\big\|_{L^{4/3}(\mathbb{R}^2)}\Big)\to 0 \text{ as } t\to 0. \tag{3.25}$$

Indeed, let $\{u_{i0}^n\}, i=1,2$, be a sequence of functions in $C_0^\infty(\mathbb{R}^2)$ satisfying $u_{i0}^n\to u_{i0}$ in $L^1(\mathbb{R}^2)$ as $n\to\infty$. By (3.18), we get

$$\begin{aligned}
&t^{1/4}\big\|e^{\mu_i t\Delta}u_{i0}^n\big\|_{L^{4/3}(\mathbb{R}^2)}\\
&\le t^{1-\frac{1}{q}}(4\pi\mu_i)^{\frac{3}{4}-\frac{1}{q}}\|u_{i0}^n\|_{L^q(\mathbb{R}^2)}\to 0 \text{ as } t\to 0 \text{ for } 1<q\le\infty,
\end{aligned}$$

and

$$\begin{aligned}
&t^{1/4}\big\|e^{\mu_i t\Delta}u_{i0}-e^{\mu_i t\Delta}u_{i0}^n\big\|_{L^{4/3}(\mathbb{R}^2)}\\
&=t^{1/4}\big\|e^{\mu_i t\Delta}(u_{i0}-u_{i0}^n)\big\|_{L^{4/3}(\mathbb{R}^2)}\\
&\le (4\pi\mu_i)^{-\frac{1}{4}}\|u_{i0}-u_{i0}^n\|_{L^1(\mathbb{R}^2)}\to 0 \text{ as } n\to\infty.
\end{aligned}$$

Therefore, for every $\delta>0$, there exist $N(\delta)\in\mathbb{N}$ and $T_\delta\in(0,\infty)$ such that

$$\begin{aligned}
&t^{1/4}\big\|e^{\mu_i t\Delta}u_{i0}\big\|_{L^{4/3}(\mathbb{R}^2)}\\
&\le t^{1/4}\big\|e^{\mu_i t\Delta}u_{i0}-e^{\mu_i t\Delta}u_{i0}^N\big\|_{L^{4/3}(\mathbb{R}^2)}+t^{1/4}\big\|e^{\mu_i t\Delta}u_{i0}^N\big\|_{L^{4/3}(\mathbb{R}^2)}\\
&\le (4\pi\mu_i)^{-\frac{1}{4}}\big\|u_{i0}^N-u_{i0}\big\|_{L^1(\mathbb{R}^2)}+t(4\pi\mu_i)^{\frac{3}{4}}\big\|u_{i0}^N\big\|_{L^\infty(\mathbb{R}^2)}<\frac{\delta}{4},
\end{aligned}$$

for all $t \in (0, T_\delta]$, $i = 1, 2$. Therefore, for every $0 < \delta < 1/(2C_B)$, there exists $T_\delta > 0$ such that $\left\|(e^{\mu_1 t\Delta}u_{10}, e^{\mu_2 t\Delta}u_{20})\right\|_{X_{T_\delta}} < \delta/2$ and by applying the Lemma 10, we have that the integral equation (3.16) has a solution $(u_1, u_2) \in X_{T_\delta}$. Moreover, this solution is such that $\|(u_1, u_2)\|_{X_{T_\delta}} < \delta$ and unique in the open ball of radius δ. The solution is in $BC\left((0, T_\delta); L^1(\mathbb{R}^2)\right)^2$, and hence a mild solution of (3.1) on $[0, T_\delta)$, since $(e^{\mu_1 t\Delta}u_{10}, e^{\mu_2 t\Delta}u_{20})$ and $B\left((u_1, u_2), (u_1, u_2)\right)$ belong to $BC\left((0, T_\delta); L^1(\mathbb{R}^2)\right)^2$. Indeed, applying (3.18), (3.19) and (3.20), we obtain

$$
\begin{aligned}
&\|u_i\|_{L^1(\mathbb{R}^2)} \\
&\leq \left\|e_{i0}^{\mu_i t\Delta}u_{i0}\right\|_{L^1(\mathbb{R}^2)} + \chi_i \left\|B_i\left((u_1, u_2), (u_1, u_2)\right)\right\|_{L^1(\mathbb{R}^2)} \\
&\leq \|u_{i0}\|_{L^1(\mathbb{R}^2)} + C_3 \int_0^t (t-s)^{-\frac{1}{2}} \left\|\left|u_i(s)A_i\left(\nabla \mathbf{K} * (a_1u_1 + a_2u_2)\right)(s)\right|\right\|_{L^1(\mathbb{R}^2)} ds \\
&\leq \|u_{i0}\|_{L^1(\mathbb{R}^2)} + C_4 \int_0^t (t-s)^{-\frac{1}{2}} \left(\|u_1(s)\|_{L^{4/3}(\mathbb{R}^2)} + \|u_2(s)\|_{L^{4/3}(\mathbb{R}^2)}\right)^2 ds \\
&\leq \|u_{i0}\|_{L^1(\mathbb{R}^2)} + C_4 \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds \, \|(u_1, u_2)\|_{X_{T_\delta}}^2 \\
&\leq \|u_{i0}\|_{L^1(\mathbb{R}^2)} + C_5 \|(u_1, u_2)\|_{X_{T_\delta}}^2, i = 1, 2.
\end{aligned}
$$

Uniqueness. Based on [16], we can prove that if $(u_1, u_2) \in X_T$ is a mild solution of (3.1) on $[0, T)$ with initial data $(u_1(0), u_2(0))$. Then

$$
t^{1/4}\left(\|u_1(t)\|_{L^{4/3}(\mathbb{R}^2)} + \|u_2(t)\|_{L^{4/3}(\mathbb{R}^2)}\right) \to 0 \text{ as } t \to 0. \tag{3.26}
$$

Indeed, for $\tau \in (0, T/2]$, the function $t \mapsto (u_1(t+\tau), u_2(t+\tau))$ is a mild solution of (3.1) on $[0, T/2) \subset [0, T-\tau)$ with initial data $(u_1(\tau), u_2(\tau))$. We rewrite the integral equation (3.15) as

$$
\begin{aligned}
u_i(t+\tau) &= e^{\mu_i t\Delta}u_i(\tau) \\
&\quad - \chi_i \int_0^t e^{\mu_i(t-s)\Delta}\nabla \cdot \left(\left(u_i(s+\tau)A_i\left(\nabla \mathbf{K} * (a_1u_1 + a_2u_2)\right)(s+\tau)\right)\right) ds \\
&= e^{\mu_i t\Delta}u_i(\tau) \\
&\quad - \chi_i \int_\tau^{t+\tau} e^{\mu_i(t+\tau-s)\Delta}\nabla \cdot \left(\left(u_i(s)A_i\left(\nabla \mathbf{K} * (a_1u_1 + a_2u_2)\right)(s)\right)\right) ds.
\end{aligned}
$$

On the one hand, by a density argument as (3.25), we have

$$
t^{1/4}\left(\left\|e^{\mu_1 t\Delta}u_1(\tau)\right\|_{L^{4/3}(\mathbb{R}^2)} + \left\|e^{\mu_2 t\Delta}u_2(\tau)\right\|_{L^{4/3}(\mathbb{R}^2)}\right) \to 0 \text{ as } t \to 0,
$$

since $u_i \in C\left([0,T)\,; L^1\left(\mathbb{R}^2\right)\right), i=1,2$. On the other hand, notice that

$$\begin{aligned}
&t^{1/4}\int_\tau^{t+\tau}\left\|e^{\mu_i(t+\tau-s)\Delta}\nabla\cdot\left((u_i(s)A_i\left(\nabla\mathbf{K}*\left(a_1u_1+a_2u_2\right)\right)(s)\right)\right\|_{L^{4/3}(\mathbb{R}^2)}ds\\
&\le C_6t^{1/4}\int_\tau^{t+\tau}\left(t+\tau-s\right)^{-\frac{3}{4}}s^{-\frac{1}{2}}ds\left\|(u_1,u_2)\right\|_{X_T}^2\\
&=C_6\left(\frac{t}{t+\tau}\right)^{1/4}\int_{\frac{\tau}{t+\tau}}^1\left(1-s\right)^{-\frac{3}{4}}s^{-\frac{1}{2}}ds\left\|(u_1,u_2)\right\|_{X_T}^2\\
&\le C_6\left(\frac{t}{t+\tau}\right)^{1/4}\int_0^1\left(1-s\right)^{-\frac{3}{4}}s^{-\frac{1}{2}}ds\left\|(u_1,u_2)\right\|_{X_T}^2\to 0 \text{ as } t\to 0, i=1,2.
\end{aligned}$$

It is follows that

$$t^{1/4}\left(\left\|u_1(t+\tau)\right\|_{L^{4/3}(\mathbb{R}^2)}+\left\|u_2(t+\tau)\right\|_{L^{4/3}(\mathbb{R}^2)}\right)\to 0 \text{ as } t\to 0. \tag{3.27}$$

Using the fact that the set $Y_i := \{u_i(\tau)\in L^1\left(\mathbb{R}^2\right); \tau\in(0,T/2]\}, i=1,2$, is precompact (then totally bounded i.e., for all $\delta>0$, there is a finite subset $Y := \{u_1(\tau_j), u_2(\tau_j)\in L^1\left(\mathbb{R}^2\right); \tau_j\in(0,T/2], j\in\{1,2,\ldots,n\}, n\in\mathbb{N}\}$ such that $Y_i\subset\bigcup_{j=1}^n B(u_i(\tau_j),\delta/4))$ in $L^1(\mathbb{R}^2)$ since $u_i\in C\left([0,T)\,;L^1\left(\mathbb{R}^2\right)\right), i=1,2$ and by a density argument as (3.25), we have that: for every $0<\delta<1/\left(2C_B\right)$, there exists $T_\delta\in(0,T/2)$ such that

$$\left\|\left(e^{\mu_1t\Delta}u_1(\tau), e^{\mu_2t\Delta}u_2(\tau)\right)\right\|_{X_{T_\delta}}<\delta/2 \text{ for all } \tau\in(0,T/2].$$

By Lemma 10 and the uniqueness condition (3.27) applied to the initial condition $\left(e^{\mu_1t\Delta}u_1(\tau), e^{\mu_2t\Delta}u_2(\tau)\right)$, we know that

$$\sup_{0<t<T_\delta}t^{1/4}\left(\left\|u_1(t+\tau)\right\|_{L^{4/3}(\mathbb{R}^2)}+\left\|u_2(t+\tau)\right\|_{L^{4/3}(\mathbb{R}^2)}\right)<\delta \text{ for all } \tau\in(0,T/2]. \tag{3.28}$$

As $\tau\to 0$ in (3.28) (with $t\in(0,T_\delta)$ being fixed), we obtain

$$t^{1/4}\left(\left\|u_1(t)\right\|_{L^{4/3}(\mathbb{R}^2)}+\left\|u_2(t)\right\|_{L^{4/3}(\mathbb{R}^2)}\right)<\delta.$$

Then, we are reduced to the uniqueness condition (3.26).

(**i**) *Mass conservation.* Fix $t>s>0$. Define $f_i:\mathbb{R}^2\to\mathbb{R}^2$ by

$$f_i(x)=e^{\mu_i(t-s)\Delta}\left(u_i(s)A_i\left(\nabla\mathbf{K}*\left(a_1u_1+a_2u_2\right)\right)(s)\right)(x), i=1,2.$$

By (3.18), (3.19) and (3.20), we have that

$$\begin{aligned}
&\left\|\left|f_i\right|\right\|_{L^1(\mathbb{R}^2)}\\
&\le\left\|\left|u_i(s)A_i\left(\nabla\mathbf{K}*\left(a_1u_1+a_2u_2\right)\right)(s)\right|\right\|_{L^1(\mathbb{R}^2)}\\
&\le\left\|u_i(s)\right\|_{L^{4/3}(\mathbb{R}^2)}\left\|\left|\left(\nabla\mathbf{K}*\left(a_1u_1+a_2u_2\right)\right)(s)\right|\right\|_{L^4(\mathbb{R}^2)}\\
&\le C_7\left\|u_i(s)\right\|_{L^{4/3}(\mathbb{R}^2)}\left(\left|a_1\right|\left\|u_1(s)\right\|_{L^{4/3}(\mathbb{R}^2)}+\left|a_2\right|\left\|u_2(s)\right\|_{L^{4/3}(\mathbb{R}^2)}\right)<\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\|\nabla \cdot f_i\|_{L^1(\mathbb{R}^2)} \\
&\leq C_8(t-s)^{-1/2} \||u_i(s)A_i\left(\nabla \mathbf{K} * (a_1u_1 + a_2u_2)\right)(s)|\|_{L^1(\mathbb{R}^2)} \\
&\leq C_9(t-s)^{-1/2} \|u_i(s)\|_{L^{4/3}(\mathbb{R}^2)} \left(|a_1| \|u_1(s)\|_{L^{4/3}(\mathbb{R}^2)} + |a_2| \|u_2(s)\|_{L^{4/3}(\mathbb{R}^2)}\right) \\
&< \infty.
\end{aligned}$$

The proposition in [45, p. 179] gives

$$\int_{\mathbb{R}^2} \nabla \cdot f_i dx = 0, i = 1, 2.$$

Moreover, recall that

$$\int_{\mathbb{R}^2} e^{\mu_i t\Delta} u_{i0} dx = \int_{\mathbb{R}^2} u_{i0} dx,$$

since $\int_{\mathbb{R}^2} G_{t,\mu_i} dx = 1$ (See [45, p. 14]).Then, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} u_i dx \\
&= \int_{\mathbb{R}^2} e^{\mu_i t\Delta} u_{i0} dx - \chi_i \int_0^t \int_{\mathbb{R}^2} \nabla \cdot e^{\mu_i(t-s)\Delta} \left(u_i(s) A_i \nabla v(s)\right) dx ds \\
&= \int_{\mathbb{R}^2} u_{i0} dx + \chi_i \int_0^t \int_{\mathbb{R}^2} \nabla \cdot f_i dx ds = \int_{\mathbb{R}^2} u_{i0} dx = \theta_i, i = 1, 2.
\end{aligned}$$

(ii) *Integrability.* From (3.15), using (3.18), (3.19), we have for any $1 \leq q \leq p \leq \infty$ that

$$\begin{aligned}
&\|u_i\|_{L^p(\mathbb{R}^2)} \\
&\leq \left\|e^{\mu_i t\Delta} u_{i0}\right\|_{L^p(\mathbb{R}^2)} + \chi_i \int_0^t \left\|e^{\mu_i(t-s)\Delta} \nabla \cdot \left(u_i(s) A_i \nabla v(s)\right)\right\|_{L^p(\mathbb{R}^2)} ds \\
&\leq (4\pi\mu_i t)^{\frac{1}{p}-1} \|u_{i0}\|_{L^1(\mathbb{R}^2)} \\
&+ C_{10}\chi_i \int_0^t (t-s)^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}} \|u_i(s) A_i \nabla v(s)\|_{L^q(\mathbb{R}^2)} ds.
\end{aligned}$$

Now we recall the following extension of Hölder's inequality: For all $f \in L^p(\mathbb{R}^2), g \in L^q(\mathbb{R}^2), 1 \leq p, q \leq \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \leq 1$, then $fg \in L^r(\mathbb{R}^2)$ and

$$\|fg\|_{L^r(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)} \|g\|_{L^q(\mathbb{R}^2)}. \tag{3.29}$$

By (3.29), we get that

$$\begin{aligned}
&\|u_i\|_{L^p(\mathbb{R}^2)} \\
&\le (4\pi\mu_i t)^{\frac{1}{p}-1}\|u_{i0}\|_{L^1(\mathbb{R}^2)} \\
&+ C_{10}\chi_i\int_0^t (t-s)^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}\|u_i(s)\|_{L^{\frac{2r}{2+r}}(\mathbb{R}^2)}\||A_i\nabla v(s)|\|_{L^r(\mathbb{R}^2)}\,ds \\
&= (4\pi\mu_i t)^{\frac{1}{p}-1}\|u_{i0}\|_{L^1(\mathbb{R}^2)} \\
&+ C_{10}\chi_i\int_0^t (t-s)^{\frac{1}{p}-\frac{2+q}{2q}}\|u_i(s)\|_{L^{\frac{2r}{2+r}}(\mathbb{R}^2)}\||(\nabla\mathbf{K}*(a_1u_1+a_2u_2))(s)|\|_{L^r(\mathbb{R}^2)}\,ds \\
&\le (4\pi\mu_i t)^{\frac{1}{p}-1}\|u_{i0}\|_{L^1(\mathbb{R}^2)} \\
&+ \frac{C_{10}\chi_i}{2\pi}\int_0^t (t-s)^{\frac{1}{p}-\frac{2+q}{2q}}\|u_i(s)\|_{L^{\frac{2r}{2+r}}(\mathbb{R}^2)}\left\|\left(\frac{1}{|x|}*(a_1u_1+a_2u_2)\right)(s)\right\|_{L^r(\mathbb{R}^2)}ds,
\end{aligned}$$

where $r=\frac{4q}{2-q}>2$(So $q<2$). Now, we use (3.20) to deduce

$$\begin{aligned}
&\|u_i\|_{L^p(\mathbb{R}^2)} \\
&\le (4\pi\mu_i t)^{\frac{1}{p}-1}\|u_{i0}\|_{L^1(\mathbb{R}^2)} \\
&+ \frac{C_{10}\chi_i}{2\pi}\int_0^t (t-s)^{\frac{1}{p}-\frac{2}{a}}\|u_i(s)\|_{L^a(\mathbb{R}^2)}\|a_1u_1(s)+a_2u_2(s)\|_{L^a(\mathbb{R}^2)}\,ds \\
&\le (4\pi\mu_i t)^{\frac{1}{p}-1}\|u_{i0}\|_{L^1(\mathbb{R}^2)} \\
&+ C_{11}\left(\sup_{0<t<T} t^{1-\frac{1}{a}}\left(\|u_1(t)\|_{L^a(\mathbb{R}^2)}+\|u_2(t)\|_{L^a(\mathbb{R}^2)}\right)\right)^2\int_0^t (t-s)^{\frac{1}{p}-\frac{2}{a}}s^{\frac{2}{a}-2}ds,
\end{aligned}$$

where $a=\frac{4q}{2+q}=\frac{2r}{2+r}<2$(Since $r=\frac{2a}{2-a}$). If $\frac{1}{p}-\frac{2}{a}+1>0$,or equivalently $a>\frac{2p}{1+p}$, we have that

$$\begin{aligned}
&t^{1-\frac{1}{p}}\|u_i\|_{L^p(\mathbb{R}^2)} \\
&\le (4\pi\mu_i)^{\frac{1}{p}-1}\|u_{i0}\|_{L^1(\mathbb{R}^2)} \\
&+ C_{12}(p,a)\left(\sup_{0<t<T} t^{1-\frac{1}{a}}\left(\|u_1(t)\|_{L^a(\mathbb{R}^2)}+\|u_2(t)\|_{L^a(\mathbb{R}^2)}\right)\right)^2.
\end{aligned}\tag{3.30}$$

Therefore, we need to choose $a\in\left(\frac{2p}{1+p},2\right)$. For $1\le p<2$, taking $a=4/3$ so that $q=1$, $r=4$, and $\frac{1}{p}-\frac{2}{a}+1=\frac{1}{p}-\frac{1}{2}>0$, (3.30) becomes

$$t^{1-\frac{1}{p}}\|u_i\|_{L^p(\mathbb{R}^2)}\le(4\pi\mu_i)^{\frac{1}{p}-1}\|u_{i0}\|_{L^1(\mathbb{R}^2)}+C_{12}\|(u_1,u_2)\|_{X_T}^2.$$

For $p=2$, taking $a=5/3$ so that $q=10/7$, $r=10$, and $\frac{1}{p}-\frac{2}{a}+1=\frac{3}{10}>0$, (3.30) becomes

$$\begin{aligned}
&t^{\frac{1}{2}}\|u_i\|_{L^2(\mathbb{R}^2)} \\
&\le (4\pi\mu_i)^{-\frac{1}{2}}\|u_{i0}\|_{L^1(\mathbb{R}^2)} \\
&+ C_{12}\left(\sup_{0<t<T} t^{\frac{2}{5}}\left(\|u_1(t)\|_{L^{5/3}(\mathbb{R}^2)}+\|u_2(t)\|_{L^{5/3}(\mathbb{R}^2)}\right)\right)^2.
\end{aligned}$$

For $2 < p < \infty$,taking $a = \frac{4p}{1+2p} < 2$ so that $q = \frac{2p}{1+p} \leq p$, $r = 4p$, and $\frac{1}{p} - \frac{2}{a} + 1 = \frac{1}{2p} > 0$, (3.30) becomes

$$\begin{aligned} & t^{1-\frac{1}{p}} \|u_i\|_{L^p(\mathbb{R}^2)} \\ & \leq (4\pi\mu_i)^{\frac{1}{p}-1} \|u_{i0}\|_{L^1(\mathbb{R}^2)} \\ & + C_{12} \left(\sup_{0<t<T} t^{\frac{2p-1}{4p}} \left(\|u_1(t)\|_{L^{\frac{4p}{1+2p}}(\mathbb{R}^2)} + \|u_2(t)\|_{L^{\frac{4p}{1+2p}}(\mathbb{R}^2)} \right) \right)^2 . \end{aligned}$$

For $p = \infty$, from (3.21) with $q = 3$, we have

$$\begin{aligned} & \||\nabla v|\|_{L^\infty(\mathbb{R}^2)} = \||(\nabla \mathbf{K} * (a_1 u_1 + a_2 u_2))(s)|\|_{L^\infty(\mathbb{R}^2)} \\ & \leq C_{13} \|a_1 u_1(s) + a_2 u_2(s)\|_{L^1(\mathbb{R}^2)}^{1/4} \|a_1 u_1(s) + a_2 u_2(s)\|_{L^3(\mathbb{R}^2)}^{3/4} \\ & \leq C_{14} \left(\|u_1(s)\|_{L^1(\mathbb{R}^2)} + \|u_2(s)\|_{L^1(\mathbb{R}^2)} \right)^{\frac{1}{4}} \left(\|u_1(s)\|_{L^3(\mathbb{R}^2)} + \|u_2(s)\|_{L^3(\mathbb{R}^2)} \right)^{\frac{3}{4}} \end{aligned}$$

Therefore

$$\begin{aligned} & \|u_i\|_{L^\infty(\mathbb{R}^2)} \\ & \leq \left\| e^{\mu_i t/2\Delta} u_i(t/2) \right\|_{L^\infty(\mathbb{R}^2)} + \chi_i \int_{t/2}^t \left\| e^{\mu_i (t-s)\Delta} \nabla \cdot (u_i(s) A_i \nabla v(s)) \right\|_{L^\infty(\mathbb{R}^2)} ds \\ & \leq (2\pi\mu_i t)^{-1} \|u_i(t/2)\|_{L^1(\mathbb{R}^2)} + C_{15}\chi_i \int_{t/2}^t (t-s)^{-\frac{1}{2}-\frac{1}{3}} \|u_i(s) A_i \nabla v(s)\|_{L^3(\mathbb{R}^2)} ds \\ & \leq (2\pi\mu_i t)^{-1} \sup_{0<s<T} \|u_i(s)\|_{L^1(\mathbb{R}^2)} \\ & + C_{15}\chi_i \int_{t/2}^t (t-s)^{\frac{1}{6}-1} \|u_i(s)\|_{L^3(\mathbb{R}^2)} \||A_i \nabla v|\|_{L^\infty(\mathbb{R}^2)} ds \\ & \leq (2\pi\mu_i t)^{-1} \sup_{0<s<T} \|u_i(s)\|_{L^1(\mathbb{R}^2)} \\ & + C_{16}\chi_i \int_{t/2}^t (t-s)^{\frac{1}{6}-1} \left(\|u_1(s)\|_{L^3(\mathbb{R}^2)} + \|u_2(s)\|_{L^3(\mathbb{R}^2)} \right)^{\frac{7}{4}} ds \\ & \leq (2\pi\mu_i t)^{-1} \sup_{0<s<T} \|u_i(s)\|_{L^1(\mathbb{R}^2)} \\ & + C_{16}\chi_i \left(\sup_{0<t<T} t^{\frac{2}{3}} \left(\|u_1(t)\|_{L^3(\mathbb{R}^2)} + \|u_2(t)\|_{L^3(\mathbb{R}^2)} \right) \right)^{7/4} \int_{t/2}^t (t-s)^{\frac{1}{6}-1} s^{-\frac{1}{6}-1} ds \end{aligned}$$

Then

$$\begin{aligned} & t \|u_i\|_{L^\infty(\mathbb{R}^2)} \\ & \leq (2\pi\mu_i)^{-1} \sup_{0<s<T} \|u_i(s)\|_{L^1(\mathbb{R}^2)} \\ & + C_{17}\chi_i \left(\sup_{0<t<T} t^{\frac{2}{3}} \left(\|u_1(t)\|_{L^3(\mathbb{R}^2)} + \|u_2(t)\|_{L^3(\mathbb{R}^2)} \right) \right)^{7/4} \end{aligned}$$

(**iii**) *Decay rates.* Following [64, Proposition 2.3.], we introduce $D^k = (-\Delta)^{k/2}$ for all real $k \geq 0$ defined by

$$D^k f = \mathcal{F}^{-1} \left[|\xi|^k \mathcal{F}[f] \right] \text{ for } f \in \mathbf{S}',$$

where $\mathbf{S}'$ is the space of tempered distributions on $\mathbb{R}^2$. Here $\mathcal{F}$ and $\mathcal{F}^{-1}$are the Fourier transform and the inverse Fourier transform in $\mathbb{R}^2$, respectively. For $f \in \mathbf{S}$, where $\mathbf{S}$ is the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^2$,

$$\mathcal{F}[f](\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)dx,$$
$$\mathcal{F}^{-1}[f](x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ix\cdot\xi} f(\xi)d\xi.$$

The heat semigroup $e^{\mu_i t\Delta}$ has the following properties (c.f. [84, Lemma 2.1, Proposition 2.1], [51, Lemma 1.1]): Let $f \in L^q(\mathbb{R}^2)$, $q \geq 1$, $k \in \mathbb{R}_+$ and $m \in \mathbb{Z}_+$. Then, there exists a constant $\tilde{C}_j = C_j(k,m,p,q,\mu_i), j = 1,2,3,$ such that, for any $1 \leq q \leq p \leq \infty$

(a) $\left\|D^k e^{\mu_i t\Delta} f\right\|_{L^p(\mathbb{R}^2)} \leq \tilde{C}_1 t^{-k/2-(1/q-1/p)} \|f\|_{L^q(\mathbb{R}^2)}$ with $k > 0$,

(b) $\left\|D^k \nabla\cdot e^{\mu_i t\Delta} F\right\|_{L^p(\mathbb{R}^2)} \leq \tilde{C}_2 t^{-k/2-1/2-(1/q-1/p)} \|F\|_{L^q(\mathbb{R}^2)}$ with $k > 0$,

(c) $\left\|\partial_t^m D^k e^{\mu_i t\Delta} f\right\|_{L^p(\mathbb{R}^2)} \leq \tilde{C}_3 t^{-k/2+m-(1/q-1/p)} \|f\|_{L^q(\mathbb{R}^2)}$ with $k + m > 0$,[1]

Let $R_j f, j = 1,2,$ be the Riesz transforms of $f \in \mathbf{S}'$ defined by

$$R_j f = \mathcal{F}^{-1}\left[i\frac{\xi_j}{|\xi|}\mathcal{F}[f]\right],\ j = 1,2,$$

The Riesz transforms R_j has the following properties (c.f. [81])

(A) R_j are bounded linear operators on $L^p(\mathbb{R}^2)$ for any $1 < p < \infty$,

(B) $(R_1^2 + R_2^2) f = -f$, $R_1 R_2 f = R_1 R_2 f$ for $f \in \mathbf{S}'$,

(C) $DR_j f = R_j Df$, $\partial_k R_j f = R_j \partial_k f$ for $f \in \mathbf{S}'$,

(D) $R_1^{l_1} R_2^{l_2} D^{|l|} f = \partial_1^{l_1}\partial_2^{l_2} f$ for $f \in \mathbf{S}'$, where $l = (l_1, l_2) \in \mathbb{Z}_+^2$ with $|l| \geq 1$.

Now, we can adapted the techniques used in [51] for the vorticity equation in $\mathbb{R}^2$ to show that the mild solution (u_1, u_2) to (3.1) on $[0,T)$ satisfy $D^k u_i \in C((0,T); L^p(\mathbb{R}^2))$ and

$$\sup_{0<t<T}(t^{1-1/p+k/2}\left\|D^k u_i\right\|_{L^p(\mathbb{R}^2)}) < \infty,\ \text{for all real } k \geq 0,\ 1 < p < \infty,\ i = 1,2. \tag{3.31}$$

We use induction for k. First we note that (**3.31**) is equivalent to (**ii**) for $k = 0$. Assuming that (**3.31**) is true for some $k \geq 0$, we prove it for $k + h$, where $0 < h < 1$. Fix such k and h, we have to prove

$$\sup_{0<t<T}(t^{1-1/p+(k+h)/2}\left\|D^{k+h} u_i\right\|_{L^p(\mathbb{R}^2)}) < \infty, \text{for any } 1 < p < \infty, i = 1,2.$$

To this end we take $\delta = 1 - \frac{1}{p} + \frac{k+h}{2} > 0$, and we note that u_i satisfies the integral equation

$$u_i(t) = \delta t^{-\delta}\int_0^t e^{\mu_i(t-s)\Delta} s^{\delta-1} u_i(s)ds - \chi_i t^{-\delta}\int_0^t \nabla\cdot e^{\mu_i(t-s)\Delta} s^{\delta}\left(u_i(s)A_i\nabla v(s)\right)ds. \tag{3.32}$$

[1]Note that $\left\|D^k\partial_t^m e^{\mu_i t\Delta} f\right\|_{L^p(\mathbb{R}^2)} = \mu_i \left\|D^k \Delta^m e^{\mu_i t\Delta} f\right\|_{L^p(\mathbb{R}^2)}$ since $\partial_t e^{\mu_i t\Delta} f = \mu_i \Delta e^{\mu_i t\Delta} f$.

This can be verified upon deducing a differential equation for $w = t^\delta u$, analogous to (3.1) but containing the term $\delta t^{-1} w$, and then convert it back to an integral equation, obtaining (3.32). Then, we have

$$D^{k+h} u_i(t) = \delta t^{-\delta} \int_0^t D^h e^{\mu_i (t-s)\Delta} s^{\delta-1} (D^k u_i(s)) ds - \chi_i t^{-\delta} \int_0^t D^h \nabla \cdot e^{\mu_i (t-s)\Delta} s^\delta D^k \left(u_i(s) A_i \nabla v(s) \right) ds.$$

So

$$\begin{aligned} &\left\| D^{k+h} u_i \right\|_{L^p(\mathbb{R}^2)} \\ &\le \delta t^{-\delta} \left\| \int_0^t D^h e^{\mu_i (t-s)\Delta} s^{\delta-1} (D^k u_i(s)) ds \right\|_{L^p(\mathbb{R}^2)} \\ &+ \chi_i t^{-\delta} \left\| \int_0^t D^h \nabla \cdot e^{\mu_i (t-s)\Delta} s^\delta D^k \left(u_i(s) A_i \nabla v(s) \right) ds \right\|_{L^p(\mathbb{R}^2)} \\ &=: I_1 + I_2. \end{aligned}$$

By induction hypothesis and **(a)**, we have

$$\begin{aligned} I_1 &= \delta t^{-\delta} \left\| \int_0^t D^h e^{\mu_i (t-s)\Delta} s^{\delta-1} (D^k u_i(s)) ds \right\|_{L^p(\mathbb{R}^2)} \\ &\le C_{18} \delta t^{-\delta} \int_0^t (t-s)^{-h/2} s^{\delta-1} \left\| (D^k u_i(s)) \right\|_{L^p(\mathbb{R}^2)} ds \\ &\le C_{18} \delta t^{-\delta} \sup_{0<t<T} (t^{1-1/p+k/2} \left\| D^k u_i \right\|_{L^p(\mathbb{R}^2)}) \int_0^t (t-s)^{-h/2} s^{h/2-1} ds \\ &\le C_{19} \delta t^{-\delta} \sup_{0<t<T} (t^{1-1/p+k/2} \left\| D^k u_i \right\|_{L^p(\mathbb{R}^2)}) \end{aligned} \tag{3.33}$$

For the second term, we begin by showing that: for $2 < p < \infty$ if $k = 0$, and $1 < p < \infty$ if $k > 0$, we have that $D^k(\nabla \mathbf{K} * u_i) \in C((0,T); L^p(\mathbb{R}^2))$ and

$$\sup_{0<t<T} (t^{1/2-1/p+k/2} \left\| D^k (\nabla \mathbf{K} * u_i) \right\|_{L^p(\mathbb{R}^2)}) < \infty, \text{ for } i = 1, 2, \tag{3.34}$$

are consequences of (**3.31**). In fact, note that $D(\partial_j \mathbf{K} * u_i) = R_j u_i$, where $j = 1, 2$ and $1 < p < \infty$. Thus, $D^k(\partial_j \mathbf{K} * u_i) = D^{k-1} R_j u_i = R_j D^{k-1} u_i$. If $k \ge 1$, (3.34) thus follows from (3.31). If $0 < k < 1$, D^{k-1} is a bounded operator on $L^p(\mathbb{R}^2)$ onto $L^q(\mathbb{R}^2)$ with $\frac{1}{q} = \frac{1}{p} - \frac{1-k}{2}$ (c.f. [81, Theorem 1. p. 119]). Then

$$\begin{aligned} \left\| D^{k-1} u_i \right\|_{L^q(\mathbb{R}^2)} &\le C_{20} \left\| u_i \right\|_{L^p(\mathbb{R}^2)} \\ &\le C_{21} t^{-1+1/p} \\ &= C_{21} t^{-1/2+1/q-k/2}. \end{aligned}$$

If $k = 0$, From (3.20) and (3.31), it follows that

$$\begin{aligned} \left\| \nabla \mathbf{K} * u_i \right\|_{L^p(\mathbb{R}^2)} &\le C_{22} \left\| u_i \right\|_{L^{\frac{2p}{2+p}}(\mathbb{R}^2)} \\ &\le C_{23} t^{-1+(2+p)/2p} \\ &= C_{23} t^{-1/2+1/p}. \end{aligned}$$

Now, from **(b)** we have that

$$I_2 = \chi_i t^{-\delta} \left\| D^h \int_0^t \nabla \cdot e^{\mu_i (t-s)\Delta} s^\delta D^k \left(u_i(s) A_i \nabla v(s)\right) ds \right\|_{L^p(\mathbb{R}^2)}$$
$$\leq C_{24} \chi_i t^{-\delta} \int_0^t (t-s)^{-1/2-h/2} s^\delta \left\| D^k \left(u_i(s) A_i \nabla v(s)\right)\right\|_{L^p(\mathbb{R}^2)} ds$$
$$\leq C_{24} \chi_i t^{-\delta}$$
$$\sum_{j=0}^k \binom{k}{j} \int_0^t (t-s)^{-1/2-h/2} s^\delta \left\| D^{k-j}(u_i(s)) A_i D^j (\nabla v(s))\right\|_{L^p(\mathbb{R}^2)} ds$$

By Holder's inequality for

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r}, r > 2$$

we obtain

$$\frac{1}{\chi_i C_{24}} t^\delta I_2$$
$$\leq \sum_{j=0}^k \binom{k}{j} \int_0^t (t-s)^{-1/2-h/2} s^\delta \left\| D^{k-j} u_i(s)\right\|_{L^q(\mathbb{R}^2)} \left\| \left| A_i D^j \nabla v(s)\right| \right\|_{L^r(\mathbb{R}^2)} ds$$
$$= \sum_{j=0}^k \binom{k}{j} \int_0^t (t-s)^{-1/2-h/2} s^\delta \left\| D^{k-j} u_i(s)\right\|_{L^q(\mathbb{R}^2)} \left\| \left| D^j \nabla v(s)\right| \right\|_{L^r(\mathbb{R}^2)} ds$$

By induction hypothesis, we have

$$\frac{1}{\chi_i C_{24}} t^\delta I_2 \leq \sum_{j=0}^k \left(\binom{k}{j} \sup_{0<t<T} (t^{1-1/q+(k-j)/2} \left\| D^{k-j} u_i\right\|_{L^q(\mathbb{R}^2)}) \right.$$
$$\sup_{0<t<T} (t^{1/2-1/r+j/2} \left\| \left| D^j \nabla v \right| \right\|_{L^r(\mathbb{R}^2)})$$
$$\left. \int_0^t (t-s)^{-1/2-h/2} s^{1-1/p+(k+h)/2} s^{-3/2+1/p-k/2} ds \right)$$
$$= \sum_{j=0}^k \left(\binom{k}{j} \sup_{0<t<T} (t^{1-1/q+(k-j)/2} \left\| D^{k-j} u_i\right\|_{L^q(\mathbb{R}^2)}) \right.$$
$$\sup_{0<t<T} (t^{1/2-1/r+j/2} \left\| \left| D^j \nabla v \right| \right\|_{L^r(\mathbb{R}^2)})$$
$$\left. \int_0^t (t-s)^{1/2-h/2-1} s^{-1/2+h/2} ds \right)$$
$$\leq C_{25} \sum_{j=0}^k \left(\binom{k}{j} \sup_{0<t<T} (t^{1-1/q+(k-j)/2} \left\| D^{k-j} u_i\right\|_{L^q(\mathbb{R}^2)}) \right.$$
$$\sup_{0<t<T} (t^{1/2-1/r+j/2} \left\| \left| D^j \nabla v \right| \right\|_{L^r(\mathbb{R}^2)}) \tag{3.35}$$

Combining (3.33) and (3.35) yields (3.31). Now, we can also apply induction for $m \in \mathbb{Z}_+$ to prove that

$$\sup_{0<t<T} (t^{1-1/p+k/2+m} \left\| \partial_t^m D^k u_i \right\|_{L^p(\mathbb{R}^2)}) < \infty, \tag{3.36}$$

for all $m \in \mathbb{Z}_+$, $k \in \mathbb{Z}_+$, $1 < p < \infty$, $i = 1, 2$. First we note that (**3.36**) is equivalent to (3.31) for $m = 0$. Assuming that (**3.36**) is true for any $k \in \mathbb{Z}_+$ and $m < N$ for some positive integer N, we can prove it for any $k \in \mathbb{Z}_+$ and $m = N$. The proof relies on direct calculation and the properties of the heat semigroup $e^{\mu_i t\Delta}$ (c.f. [84, Theorem 3.3]). At length, by (**3.36**) and the properties of the Riesz transforms R_j, we have proved (**iii**).

(iv) *Regularity.* By (**iii**) which implies that $(u_1, u_2) \in (C^{2,1}(\mathbb{R}^2 \times (0,T))^2$ and the fact that $\nabla\cdot$ and $e^{\mu_i t\Delta}$ commute on $C^1(\mathbb{R}^2;\mathbb{R}^2) \cap L^1(\mathbb{R}^2;\mathbb{R}^2)$, we can conclude that (u_1, u_2) is a classical solution of (3.1) in $\mathbb{R}^2 \times (0,T)$. It remains to show that $u_i, i = 1, 2$, satisfies the initial condition $u_i \to u_{i0}$ in $L^1(\mathbb{R}^2)$ as $t \to 0$. So, it is suffices to show that

$$\begin{aligned}
&\|u_i(t) - u_{i0}\|_{L^1(\mathbb{R}^2)} \\
&\leq \left\|e^{\mu_i t\Delta}u_{i0} - u_{i0}\right\|_{L^1(\mathbb{R}^2)} + \chi_i \left\|B_i\left((u_1, u_2), (u_1, u_2)\right)\right\|_{L^1(\mathbb{R}^2)} \to 0,
\end{aligned}$$

as $t \to 0, i = 1, 2$. But notice that this is true since

$$t^{1/4}\left(\|u_1^\varepsilon(t)\|_{L^{4/3}(\mathbb{R}^2)} + \|u_2^\varepsilon(t)\|_{L^{4/3}(\mathbb{R}^2)}\right) \to 0 \text{ as } t \to 0.$$

(**v**) We define the Banach space Y by

$$Y := \left\{(u_1(\cdot,t), u_2(\cdot,t)) \in C\left([0,T_0]\,; L^1\left(\mathbb{R}^2\right) \cap H^1(\mathbb{R}^2)\right)^2\right\},$$

with norm $\|\cdot\|_Y$

$$\begin{aligned}
&\|(u_1, u_2)\|_Y \\
&:= \sup_{0<t<T_0}\left(\|u_1(\cdot,t)\|_{L^1(\mathbb{R}^2)} + \|u_2(\cdot,t)\|_{L^1(\mathbb{R}^2)}\right) \\
&+ \sup_{0<t<T_0}\left(\|u_1(\cdot,t)\|_{H^1(\mathbb{R}^2)} + \|u_2(\cdot,t)\|_{H^1(\mathbb{R}^2)}\right).
\end{aligned}$$

Since $\nabla \cdot A_i\left(\nabla\mathbf{K} * (a_1 w_1 + a_2 w_2)\right) = -\cos\alpha\,(a_1 w_1 + a_2 w_2)$, the bilinear form $B_i, i = 1, 2$, defined by (3.17) is rewritten as

$$\begin{aligned}
&B_i\left((u_1, u_2), (w_1, w_2)\right)(t) \\
&= \int_0^t e^{\mu_i(t-s)\Delta}\nabla \cdot \left(u_i(s)A_i\left(\nabla\mathbf{K} * (a_1 w_1 + a_2 w_2)\right)(s)\right) ds \\
&= \int_0^t e^{\mu_i(t-s)\Delta}\left(\nabla u_i(s) \cdot A_i\left(\nabla\mathbf{K} * (a_1 w_1 + a_2 w_2)\right)(s)\right) ds \\
&+ \int_0^t e^{\mu_i(t-s)\Delta}u_i(s)\nabla \cdot \left(A_i\left(\nabla\mathbf{K} * (a_1 w_1 + a_2 w_2)\right)(s)\right) ds \\
&= \int_0^t e^{\mu_i(t-s)\Delta}\left(\nabla u_i(s) \cdot A_i\left(\nabla\mathbf{K} * (a_1 w_1 + a_2 w_2)\right)(s)\right) ds \\
&- \cos\alpha \int_0^t e^{\mu_i(t-s)\Delta}u_i(s)\left(a_1 w_1 + a_2 w_2\right)(s) ds \\
&=: B_{i1}\left((u_1, u_2), (w_1, w_2)\right)(t) + B_{i2}\left((u_1, u_2), (w_1, w_2)\right)(t),
\end{aligned}$$

for $i = 1, 2$. By (3.18) and (3.19), we obtain

$$\begin{aligned}&\left\|B_{i1}\left((u_1,u_2),(w_1,w_2)\right)(t)\right\|_{L^2(\mathbb{R}^2)}\\&\leq \int_0^t \left\|\left|\nabla u_i(s)\cdot A_i\left(\nabla \mathbf{K} * (a_1w_1+a_2w_2)\right)(s)\right|\right\|_{L^2(\mathbb{R}^2)} ds,\end{aligned}$$

and

$$\begin{aligned}&\left\|\nabla B_{i1}\left((u_1,u_2),(w_1,w_2)\right)(t)\right\|_{L^2(\mathbb{R}^2)}\\&\leq C_{26}\int_0^t (t-s)^{-1/2}\left\|\left|\nabla u_i(s)\cdot A_i\left(\nabla \mathbf{K} * (a_1w_1+a_2w_2)\right)(s)\right|\right\|_{L^2(\mathbb{R}^2)} ds.\end{aligned}$$

Observing that, by Hölder's inequality and (3.21) with $q \in (2,\infty)$, we have that

$$\begin{aligned}&\left\|\left|\nabla u_i(s)\cdot A_i\left(\nabla \mathbf{K} * (a_1w_1+a_2w_2)\right)(s)\right|\right\|_{L^2(\mathbb{R}^2)}\\&\leq \left\|\nabla u_i(s)\right\|_{L^2(\mathbb{R}^2)}\left\|\left|A_i\left(\nabla \mathbf{K} * (a_1w_1+a_2w_2)\right)(s)\right|\right\|_{L^\infty(\mathbb{R}^2)}\\&\leq \frac{1}{2\pi}\left\|\nabla u_i(s)\right\|_{L^2(\mathbb{R}^2)}\left\|1/\left|x\right| * (a_1w_1+a_2w_2)(s)\right\|_{L^\infty(\mathbb{R}^2)}\\&\leq \frac{C_{27}}{2\pi}\left\|\nabla u_i(s)\right\|_{L^2(\mathbb{R}^2)}\left\|(a_1w_1+a_2w_2)(s)\right\|_{L^1(\mathbb{R}^2)}^{\frac{q-2}{2(q-1)}}\left\|(a_1w_1+a_2w_2)(s)\right\|_{L^q(\mathbb{R}^2)}^{\frac{q}{2(q-1)}}.\end{aligned}$$

Now, we recall the following Gagliardo–Nirenberg inequality (See [45, p. 190.]): For all $g \in L^1(\mathbb{R}^2)\cap H^1(\mathbb{R}^2)$, there exist a constant $C = C(q), q \geq 1$ such that

$$\left\|g\right\|_{L^q(\mathbb{R}^2)} \leq C\left\|g\right\|_{L^1(\mathbb{R}^2)}^{1/q}\left\|\nabla g\right\|_{L^2(\mathbb{R}^2)}^{1-1/q}, \tag{3.37}$$

letting $g = a_1w_1 + a_2w_2$, we get

$$\begin{aligned}&\left\|\left|\nabla u_i(s)\cdot A_i\left(\nabla \mathbf{K} * (a_1w_1+a_2w_2)\right)(s)\right|\right\|_{L^2(\mathbb{R}^2)}\\&\leq C_{28}\left\|\nabla u_i(s)\right\|_{L^2(\mathbb{R}^2)}\left\|(a_1w_1+a_2w_2)(s)\right\|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}}\left\|(a_1\nabla w_1+a_2\nabla w_2)(s)\right\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}.\end{aligned}$$

Using the Young's inequality for products, we have that

$$\begin{aligned}&\left\|\left|\nabla u_i(s)\cdot A_i\left(\nabla \mathbf{K} * (a_1w_1+a_2w_2)\right)(s)\right|\right\|_{L^2(\mathbb{R}^2)}\\&\leq \frac{C_{28}}{2}\left\|\nabla u_i(s)\right\|_{L^2(\mathbb{R}^2)}\left\|(a_1w_1+a_2w_2)(s)\right\|_{L^1(\mathbb{R}^2)}\\&+\frac{C_{28}}{2}\left\|\nabla u_i(s)\right\|_{L^2(\mathbb{R}^2)}\left\|(a_1\nabla w_1+a_2\nabla w_2)(s)\right\|_{L^2(\mathbb{R}^2)}\\&\leq \frac{C_{28}\max\{\left|a_1\right|,\left|a_2\right|\}}{2}\left\|(u_1,u_2)\right\|_Y\left\|(w_1,w_2)\right\|_Y\\&= C_{29}\left\|(u_1,u_2)\right\|_Y\left\|(w_1,w_2)\right\|_Y,\end{aligned}$$

Therefore, we obtain

$$\left\|B_{i1}\left((u_1,u_2),(w_1,w_2)\right)(t)\right\|_{L^2(\mathbb{R}^2)} \leq C_{29}t\left\|(u_1,u_2)\right\|_Y\left\|(w_1,w_2)\right\|_Y,$$

and

$$\left\|\left|\nabla B_{i1}\left((u_1,u_2),(w_1,w_2)\right)(t)\right|\right\|_{L^2(\mathbb{R}^2)} \leq C_{30}t^{1/2}\left\|(u_1,u_2)\right\|_Y\left\|(w_1,w_2)\right\|_Y.$$

Now, by (3.18) and (3.19), we obtain

$$\begin{aligned}&\|B_{i2}\left((u_1,u_2),(w_1,w_2)\right)(t)\|_{L^2(\mathbb{R}^2)}\\&\le|\cos\alpha|\int_0^t\|u_i(s)\left(a_1w_1+a_2w_2\right)(s)\|_{L^2(\mathbb{R}^2)}\,ds,\end{aligned}$$

and

$$\begin{aligned}&\||\nabla B_{i2}\left((u_1,u_2),(w_1,w_2)\right)(t)|\|_{L^2(\mathbb{R}^2)}\\&\le|\cos\alpha|\,C_{31}\int_0^t(t-s)^{-1/2}\|u_i(s)\left(a_1w_1+a_2w_2\right)(s)\|_{L^2(\mathbb{R}^2)}\,ds.\end{aligned}$$

By Hölder's inequality, we have that

$$\begin{aligned}&\|u_i(s)\left(a_1w_1+a_2w_2\right)(s)\|_{L^2(\mathbb{R}^2)}\\&=\left(\int_{\mathbb{R}^2}|u_i(s)|^2\left|\left(a_1w_1+a_2w_2\right)(s)\right|^2dx\right)^{1/2}\\&\le\left(\left\||u_i(s)|^2\right\|_{L^2(\mathbb{R}^2)}\left\|\left|\left(a_1w_1+a_2w_2\right)(s)\right|^2\right\|_{L^2(\mathbb{R}^2)}\right)^{1/2}\\&=\|u_i(s)\|_{L^4(\mathbb{R}^2)}\|\left(a_1w_1+a_2w_2\right)(s)\|_{L^4(\mathbb{R}^2)}.\end{aligned}$$

Using the Gagliardo–Nirenberg inequality (3.37) and Young's inequality for products yields that

$$\begin{aligned}&\|u_i(s)\left(a_1w_1+a_2w_2\right)(s)\|_{L^2(\mathbb{R}^2)}\\&\le C_{32}\|u_i(s)\|_{L^1(\mathbb{R}^2)}^{1/4}\|\nabla u_i(s)\|_{L^2(\mathbb{R}^2)}^{3/4}\\&\|\left(a_1w_1+a_2w_2\right)(s)\|_{L^1(\mathbb{R}^2)}^{1/4}\|\left(a_1\nabla w_1+a_2\nabla w_2\right)(s)\|_{L^2(\mathbb{R}^2)}^{3/4}\\&\le\frac{3C_{32}\max\{|a_1|,|a_2|\}}{4}\|(u_1,u_2)\|_Y\|(w_1,w_2)\|_Y\\&=C_{33}\|(u_1,u_2)\|_Y\|(w_1,w_2)\|_Y\end{aligned}$$

Therefore,

$$\begin{aligned}&\|B_{i2}\left((u_1,u_2),(w_1,w_2)\right)(t)\|_{L^2(\mathbb{R}^2)}\\&\le|\cos\alpha|\,C_{33}t\|(u_1,u_2)\|_Y\|(w_1,w_2)\|_Y,\end{aligned}$$

and

$$\begin{aligned}&\||\nabla B_{i2}\left((u_1,u_2),(w_1,w_2)\right)(t)|\|_{L^2(\mathbb{R}^2)}\\&\le|\cos\alpha|\,C_{34}t^{1/2}\|(u_1,u_2)\|_Y\|(w_1,w_2)\|_Y.\end{aligned}$$

Hence

$$\begin{aligned}&\|B\left((u_1,u_2),(w_1,w_2)\right)(t)\|_{H^1(\mathbb{R}^2)}\\&\le C_{35}(t+t^{1/2})\|(u_1,u_2)\|_Y\|(w_1,w_2)\|_Y.\end{aligned}\tag{3.38}$$

On the other hand, by (3.19),(3.20), Gagliardo–Nirenberg inequality (3.37) and Young's inequality for products, we get

$$\begin{aligned}
&\|B_i\left((u_1,u_2),(w_1,w_2)\right)\|_{L^1(\mathbb{R}^2)}\\
&\le C_{36}\int_0^t (t-s)^{-\frac{1}{2}}\,\||u_i(s)A_i\left(\nabla\mathbf{K}*(a_1w_1+a_2w_2)\right)(s)|\|_{L^1(\mathbb{R}^2)}\,ds\\
&\le C_{37}\int_0^t (t-s)^{-\frac{1}{2}}\,\|u_i(s)\|_{L^{4/3}(\mathbb{R}^2)}\left(\|w_1(s)\|_{L^{4/3}(\mathbb{R}^2)}+\|w_2(s)\|_{L^{4/3}(\mathbb{R}^2)}\right)ds\\
&\le C_{37}\int_0^t (t-s)^{-\frac{1}{2}}\,s^{-\frac{1}{2}}ds\,\sup_{0<s<t} s^{1/4}\,\|u_i(s)\|_{L^{4/3}(\mathbb{R}^2)}\\
&\sup_{0<s<t} s^{1/4}\left(\|w_1(s)\|_{L^{4/3}(\mathbb{R}^2)}+\|w_2(s)\|_{L^{4/3}(\mathbb{R}^2)}\right)\\
&= C_{38}\sup_{0<s<t} s^{1/4}\,\|u_i(s)\|_{L^{4/3}(\mathbb{R}^2)}\sup_{0<s<t} s^{1/4}\left(\|w_1(s)\|_{L^{4/3}(\mathbb{R}^2)}+\|w_2(s)\|_{L^{4/3}(\mathbb{R}^2)}\right)\\
&\le C_{39}t^{1/2}\,\|(u_1,u_2)\|_Y\,\|(w_1,w_2)\|_Y\,.
\end{aligned}$$

Then

$$\begin{aligned}
&\|B\left((u_1,u_2),(w_1,w_2)\right)\|_{L^1(\mathbb{R}^2)}\\
&\le C_{40}t^{1/2}\,\|(u_1,u_2)\|_Y\,\|(w_1,w_2)\|_Y\,.
\end{aligned} \tag{3.39}$$

From (3.38) and (3.39) we deduce

$$\begin{aligned}
&\|B\left((u_1,u_2),(w_1,w_2)\right)\|_{L^1(\mathbb{R}^2)\cap H^1(\mathbb{R}^2)}\\
&\le C_{41}(t+t^{1/2})\,\|(u_1,u_2)\|_Y\,\|(w_1,w_2)\|_Y\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|B\left((u_1,u_2),(w_1,w_2)\right)\|_Y\\
&\le C_{41}(T_0+T_0^{1/2})\,\|(u_1,u_2)\|_Y\,\|(w_1,w_2)\|_Y\,,
\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}
&\|B\left((u_1,u_2),(u_1,u_2)\right)-B\left((w_1,w_2),(w_1,w_2)\right)\|_Y\\
&\le C_{41}(T_0+T_0^{1/2})\left(\|(u_1,u_2)\|_Y+\|(w_1,w_2)\|_Y\right)\\
&\|(u_1,u_2)-(w_1,w_2)\|_Y\,.
\end{aligned} \tag{3.41}$$

Now, we consider the closed set

$$S:=\left\{(u_1,u_2)\in Y,(u_1(\cdot,0),u_2(\cdot,0))=(u_{10},u_{20})\ \text{and}\ \|(u_1,u_2)\|_Y\le R\right\},$$

where

$$R:=\|u_{10}\|_{L^1(\mathbb{R}^2)}+\|u_{20}\|_{L^1(\mathbb{R}^2)}+\|u_{10}\|_{H^1(\mathbb{R}^2)}+\|u_{20}\|_{H^1(\mathbb{R}^2)}+1.$$

For $(u_1,u_2)\in S$ we set

$$\Phi\left((u_1,u_2)\right):=\left(e^{\mu_1 t\Delta}u_{10},e^{\mu_2 t\Delta}u_{20}\right)-B\left((u_1,u_2),(u_1,u_2)\right)(t),\text{for } t\in(0,T)\,.$$

It is clear that $\left(e^{\mu_1 t\Delta}u_{10}, e^{\mu_2 t\Delta}u_{20}\right) \in Y$ due to (3.18) and the fact that $u_{10}, u_{20} \in L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$. In fact,

$$\begin{aligned} &\left\| \left(e^{\mu_1 t\Delta}u_{10}, e^{\mu_2 t\Delta}u_{20}\right) \right\|_Y \\ &\leq \|u_{10}\|_{L^1(\mathbb{R}^2)} + \|u_{20}\|_{L^1(\mathbb{R}^2)} + \|u_{10}\|_{H^1(\mathbb{R}^2)} + \|u_{20}\|_{H^1(\mathbb{R}^2)} = R - 1. \end{aligned} \tag{3.42}$$

Then, we have that Φ maps S into Y. Taking $T_0 \in (0,T)$ small enough small enough such that

$$C_{41}(T_0 + T_0^{1/2})R^2 \leq 1, \text{ and } 2C_{41}(T_0 + T_0^{1/2})R < 1/2,$$

we get from (3.42),(3.40) and (3.41) that for $(u_1,u_2),(w_1,w_2) \in S$ holds

$$\|\Phi\left((u_1,u_2)\right)\|_Y \leq \left\| \left(e^{\mu_1 t\Delta}u_{10}, e^{\mu_2 t\Delta}u_{20}\right) \right\|_Y + \|B\left((u_1,u_2),(u_1,u_2)\right)\|_Y \leq R.$$

and

$$\|\Phi\left((u_1,u_2)\right) - \Phi\left((w_1,w_2)\right)\|_Y < \frac{1}{2}\|(u_1,u_2) - (w_1,w_2)\|_Y .$$

Hence the contraction mapping theorem ensures that the integral equation (3.16) has a unique solution $(\hat{u}_1,\hat{u}_2) \in S$ on $[0,T_0]$. By the fact that $\hat{u}_i \in C\left([0,T_0]\,; L^1\left(\mathbb{R}^2\right) \cap H^1(\mathbb{R}^2)\right), i = 1,2,$ and Gagliardo–Nirenberg inequality (3.37), we see $\hat{u}_i \in C\left([0,T_0]\,; L^1\left(\mathbb{R}^2\right) \cap L^{4/3}(\mathbb{R}^2)\right), i = 1,2,$ which implies that $(\hat{u}_1,\hat{u}_2)$ is a mild solution of (3.1). By uniqueness we have $(u_1,u_2) = (\hat{u}_1,\hat{u}_2)$ on $[0,T_0]$, and hence $(u_1,u_2) \in BC\left([0,T)\,; H^1(\mathbb{R}^2)\right)$.

(vi) Let $\{u_{i0}^n\}, i = 1,2,$ be a sequence of nonnegative functions in $C_0^\infty(\mathbb{R}^2)$ satisfying $u_{i0}^n \to u_{i0}$ in $L^1(\mathbb{R}^2)$ as $n \to \infty$. Notice that, using (3.18), we have that

$$\begin{aligned} &\sup_{0<t<T} t^{1/4}\left\|e^{\mu_i t\Delta}u_{i0}^n\right\|_{L^{4/3}(\mathbb{R}^2)} \\ &\leq \sup_{0<t<T} t^{1/4}\left\|e^{\mu_i t\Delta}u_{i0}\right\|_{L^{4/3}(\mathbb{R}^2)} + \sup_{0<t<T} t^{1/4}\left\|e^{\mu_i t\Delta}\left(u_{i0}^n - u_{i0}\right)\right\|_{L^{4/3}(\mathbb{R}^2)} \\ &\leq \sup_{0<t<T} t^{1/4}\left\|e^{\mu_i t\Delta}u_{i0}\right\|_{L^{4/3}(\mathbb{R}^2)} + (4\pi\mu_i t)^{-\frac{1}{4}}\left\|u_{i0}^n - u_{i0}\right\|_{L^1(\mathbb{R}^2)}, i = 1,2. \end{aligned}$$

Therefore, for every $\delta > 0$,there is $T_\delta \in (0,T)$ such that, for sufficiently large n, we have that

$$\sup_{0<t<T} t^{1/4}\left(\left\|e^{\mu_1 t\Delta}u_{10}^n\right\|_{L^{4/3}(\mathbb{R}^2)} + \left\|e^{\mu_2 t\Delta}u_{20}^n\right\|_{L^{4/3}(\mathbb{R}^2)}\right) < \delta$$

Now, we can apply the lemma 10 to get that there is $T_{\delta^*} \in (0,T), \delta^* < 1/(4C_B)$ such that, for sufficiently large n, the system (3.1), with initial data (u_{10}^n, u_{20}^n), has a mild solution $(u_1^n,u_2^n) \in C\left((0,T_{\delta^*})\,; L^1\left(\mathbb{R}^2\right)\right)^2 \cap C((0,T_{\delta^*})\,;\, L^{4/3}\left(\mathbb{R}^2\right))^2$. Moreover, this solution is such that

$$\begin{aligned} &\sup_{0<t<T_{\delta^*}} t^{1/4}\left(\|u_1^n(t) - u_1(t)\|_{L^{4/3}(\mathbb{R}^2)} + \|u_2^n(t) - u_2(t)\|_{L^{4/3}(\mathbb{R}^2)}\right) \\ &\leq (1 - 4C_B\delta^*)^{-1} \\ &\sup_{0<t<T_{\delta^*}} t^{1/4}\left(\left\|e^{\mu_1 t\Delta}\left(u_{10}^n - u_{10}\right)\right\|_{L^{4/3}(\mathbb{R}^2)} + \left\|e^{\mu_2 t\Delta}\left(u_{20}^n - u_{20}\right)\right\|_{L^{4/3}(\mathbb{R}^2)}\right) \\ &\leq (1 - 4C_B\delta^*)^{-1} \\ &\left((4\pi\mu_1)^{-\frac{1}{4}}\left\|u_{10}^n - u_{10}\right\|_{L^1(\mathbb{R}^2)} + (4\pi\mu_2)^{-\frac{1}{4}}\left\|u_{20}^n - u_{20}\right\|_{L^1(\mathbb{R}^2)}\right). \end{aligned}$$

This implies that un $u_i^n \to u_i$ in $C\left((0,T_{\delta^*});L^{4/3}(\mathbb{R}^2)\right)$ as $n\to\infty, i=1,2$. By item (**iv**) we known that each (u_1^n,u_2^n) is a classical solution of

$$\begin{array}{ll} \partial_t u_1^n = \mu_1 \Delta u_1^n - \chi_1 \nabla\cdot(u_1^n A_1 \nabla v^n), & x\in\mathbb{R}^2, t\in(0,T_{\delta^*}),\\ \partial_t u_2^n = \mu_2 \Delta u_2^n - \chi_2 \nabla\cdot(u_2^n A_2 \nabla v^n), & x\in\mathbb{R}^2, t\in(0,T_{\delta^*}),\\ \nabla v^n := \nabla\mathbf{K} * (a_1u_1^n + a_2u_2^n), & x\in\mathbb{R}^2, t\in(0,T_{\delta^*}),\\ u_1^n(x,0)=u_{10}^n(x), u_2^n(x,0)=u_{20}^n(x), & x\in\mathbb{R}^2. \end{array} \tag{3.43}$$

Multiplying the first equation of system (3.43) by $(u_1^n)^- := \max\{0,-u_1^n\}$, integrating over $\mathbb{R}^2$ and integrating by parts

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\left|(u_1^n)^-\right|^2dx\\ &=\mu_1\int_{\mathbb{R}^2}(u_1^n)^-\triangle u_1^n dx - \chi_1\int_{\mathbb{R}^2}(u_1^n)^-\nabla\cdot(u_1^nA_1\nabla v^n)dx\\ &=-\mu_1\int_{\mathbb{R}^2}\left|\nabla(u_1^n)^-\right|^2dx + \chi_1\int_{\mathbb{R}^2}\nabla(u_1^n)^-\cdot(u_1^nA_1\nabla v^n)dx\\ &=-\mu_1\int_{\mathbb{R}^2}\left|\nabla(u_1^n)^-\right|^2dx + \chi_1\int_{\mathbb{R}^2}u_1^n\nabla(u_1^n)^-\cdot(A_1\nabla v^n)dx\\ &=-\mu_1\int_{\mathbb{R}^2}\left|\nabla(u_1^n)^-\right|^2dx + \frac{\chi_1}{2}\int_{\mathbb{R}^2}\nabla\left|(u_1^n)^-\right|^2\cdot(A_1\nabla v^n)dx\\ &=-\mu_1\int_{\mathbb{R}^2}\left|\nabla(u_1^n)^-\right|^2dx - \frac{\chi_1}{2}\int_{\mathbb{R}^2}\left|(u_1^n)^-\right|^2\nabla\cdot(A_1\nabla v^n)dx. \end{aligned}$$

Using the identity $\nabla\cdot A_1\nabla v^n = \cos\alpha\Delta v^n$,we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\left|(u_1^n)^-\right|^2dx\\ &=-\mu_1\int_{\mathbb{R}^2}\left|\nabla(u_1^n)^-\right|^2dx + \frac{\chi_1\cos\alpha}{2}\int_{\mathbb{R}^2}\left|(u_1^n)^-\right|^2(-\Delta v^n)dx. \end{aligned}$$

Note that by Hölder's inequality, we get

$$\begin{aligned} &\int_{\mathbb{R}^2}\left|(u_1^n)^-\right|^2\left|(a_1u_1^n+a_2u_2^n)(s)\right|dx\\ &\le \left\|\left|(u_1^n)^-\right|^2\right\|_{L^2(\mathbb{R}^2)}\left\|(a_1u_1^n+a_2u_2^n)\right\|_{L^2(\mathbb{R}^2)}\\ &=\left\|\left|(u_1^n)^-\right|\right\|^2_{L^4(\mathbb{R}^2)}\left\|a_1u_1^n+a_2u_2^n\right\|_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{3.44}$$

At this stage, we use item (**v**) to claim that $u_i^n\in BC\left([0,T_{\delta^*}];H^1(\mathbb{R}^2)\right), i = 1,2$, and

$$\sup_{0<t<T_{\delta^*}}\left(\|u_1^n(t)\|_{L^2(\mathbb{R}^2)}+\|u_2^n(t)\|_{L^2(\mathbb{R}^2)}\right)\le C_{42},$$

for some constant $C_{42}=C(n)$ together with (3.44) to derive

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^2}\left|(u_1^n)^-\right|^2dx\\ &\le -2\mu_1\int_{\mathbb{R}^2}\left|\nabla(u_1^n)^-\right|^2dx\\ &+\chi_1\left|\cos\alpha\right|\max\{|a_1|,|a_2|\}C_{42}\left\|\left|(u_1^n)^-\right|\right\|^2_{L^4(\mathbb{R}^2)}. \end{aligned}$$

Notice that $(u_1^n)^- \in C([0,T_{\delta^*}], H^1(\mathbb{R}^2))$ since $u_1^n \in C([0,T_{\delta^*}]; H^1(\mathbb{R}^2))$ (See [40, Theorem 4 p. 130.]). Now, we recall the following Interpolation inequality of Gagliardo-Nirenberg-Sobolev (See [45, p. 190.]): There exists a constant $C_{GNB}^{(4)}$ such that

$$\|w\|_{L^4(\mathbb{R}^2)} \le C_{GNB}^{(4)} \|w\|_{L^2(\mathbb{R}^2)}^{1/2} \|\nabla w\|_{L^2(\mathbb{R}^2)}^{1/2}, \text{ for all } w \in H^1(\mathbb{R}^2), \tag{3.45}$$

letting $w = (u_1^n)^-$, we get

$$\left\|(u_1^n)^-\right\|_{L^4(\mathbb{R}^2)}^2 \le C_{42} \left\|(u_1^n)^-\right\|_{L^2(\mathbb{R}^2)} \left\|\nabla (u_1^n)^-\right\|_{L^2(\mathbb{R}^2)}.$$

In consequence

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} \left|(u_1^n)^-\right|^2 dx \\
&\le -2\mu_1 \int_{\mathbb{R}^2} \left|\nabla (u_1^n)^-\right|^2 dx \\
&+ \frac{C_{43}}{\sqrt{2\mu_1}} \left(\int_{\mathbb{R}^2} \left|(u_1^n)^-\right|^2 dx\right)^{1/2} \left(2\mu_1 \int_{\mathbb{R}^2} \left|\nabla (u_1^n)^-\right|^2 dx\right)^{1/2} \\
&\le -\mu_1 \int_{\mathbb{R}^2} \left|\nabla (u_1^n)^-\right|^2 dx + \frac{(C_{43})^2}{4\mu_1} \left(\int_{\mathbb{R}^2} \left|(u_1^n)^-\right|^2 dx\right) \\
&\le C_{44} \left(\int_{\mathbb{R}^2} \left|(u_1^n)^-\right|^2 dx\right).
\end{aligned}$$

Integrating the last inequality

$$\int_{\mathbb{R}^2} \left|(u_1^n)^-\right|^2 dx \le \left(\int_{\mathbb{R}^2} \left|(u_1^n(x,0))^-\right|^2 dx\right) e^{C_{44}t} = 0,$$

and therefore $(u_1^n)^- = 0$ on $[0,T_{\delta^*}] \times \mathbb{R}^2$. Hence, $u_1^n \ge 0$ on $[0,T_{\delta^*}] \times \mathbb{R}^2$, which implies $u_1 \ge 0$ on $[0,T_{\delta^*}] \times \mathbb{R}^2$. From

$$\int_{\mathbb{R}^2} u_1(x,t)dx = \theta_1 > 0, \text{ for all } t \in [0,T), \tag{3.46}$$

we have that $u_1(\cdot,t_0) \not\equiv 0, u_1(x,t_0) \ge 0$ for any $t_0 \in [0,T_{\delta^*})$. Similarly, we also obtain that $u_2(\cdot,t_0) \not\equiv 0, u_2(x,t_0) \ge 0$ for any $t_0 \in [0,T_{\delta^*})$. Since $(u_1,u_2) \in BC([t_0,T) \times \mathbb{R}^2)^2, i = 1,2$, for $t_0 \in (0,T_{\delta^*})$[2] and is a classical solution of

$$\begin{aligned}
\partial_t u_1 &= \mu_1 \Delta u_1 - \chi_1 (A_1 \nabla v) \cdot \nabla u_1 - \chi_1 \cos\alpha \Delta v u_1, \quad x \in \mathbb{R}^2, t \in (t_0,T), \\
\partial_t u_2 &= \mu_2 \Delta u_2 - \chi_2 (A_2 \nabla v) \cdot \nabla u_2 - \chi_2 \cos\beta \Delta v u_2, \quad x \in \mathbb{R}^2, t \in (t_0,T),
\end{aligned}$$

and $|\nabla v|, \Delta v \in BC([t_0,T) \times \mathbb{R}^2)$[3],the strong maximum principle (See [39, Theorem 12 p. 376.]) ensures that $u_i(x,t) > 0, i = 1,2$, on $(t_0,T) \times \mathbb{R}^2$. Indeed, notice that the change of variable $w_1 = u_1 e^{-\lambda t}$, with

$$\lambda := \chi_1 |\cos\alpha| \|\Delta v\|_{L^\infty([t_0,T)\times\mathbb{R}^2)} + 1.$$

---

[2]By item (**ii**), we have that $\|u_1(t)\|_{L^\infty(\mathbb{R}^2)} + \|u_2(t)\|_{L^\infty(\mathbb{R}^2)} \le Ct^{-1} \le Ct_0^{-1}, t \in [t_0,T)$.

[3]Notice that

$\||\nabla v(t)|\|_{L^\infty(\mathbb{R}^2)} \le C \|a_1u_1(t) + a_2u_2(t)\|_{L^1(\mathbb{R}^2)}^{1/4} \|a_1u_1(t) + a_2u_2(t)\|_{L^3(\mathbb{R}^2)}^{3/4}$

$\le Ct_0^{-1/2}, t \in [t_0,T)$,

and

$\|\Delta v(t)\|_{L^\infty(\mathbb{R}^2)} \le \|(a_1u_1 + a_2u_2)(t)\|_{L^\infty(\mathbb{R}^2)} \le Ct_0^{-1}, t \in [t_0,t)$.

Then w_1 satisfies

$$\partial_t w_1 = \mu_1 \Delta w_1 - \chi_1 (A_1 \nabla v) \cdot \nabla w_1 - (\chi_1 \cos \alpha \Delta v + \lambda) w_1,$$

for $x \in \mathbb{R}^2, t \in (t_0, T)$, and

$$\chi_1 \cos \alpha \Delta v(x,t) + \lambda = \chi_1 \left(\cos \alpha \Delta v(x,t) + |\cos \alpha| \, \|\Delta v\|_{L^\infty([t_0,T)\times\mathbb{R}^2)} \right) + 1 \\ =: c(x,t) \geq 0.$$

We assume that u_1 has a negative value at $(x_1, t_1) \in \mathbb{R}^2 \times (t_0, T)$, and we will arrive at a contradiction. By definition of w_1, we have that $-a = w_1(x_1, t_1) < 0$. Notice that if there is a point (x_2, t_2) in $\mathbb{R}^2 \times (t_0, t_1]$ at which $\inf_{\mathbb{R}^2 \times (t_0,t_1]} w_1 \leq -a < 0$ is attained, then w_1 is a negative constant on $B_r(0) \times (t_0, t_2]$, for all $r > |x_2|$, which implies that w_1 is a negative constant on $\mathbb{R}^2 \times (t_0, t_2]$ and this contradicts (3.46). Unfortunately, since $\mathbb{R}^2$ is unbounded, we do not know whether there exists a point at which $\inf_{\mathbb{R}^2 \times (t_0,t_1]} w_1$ is attained. Therefore, we use the following trick presented in ([45, p. 56.]). Let $b > 0$ and $\delta > 0$ be constants to be determined later, and set

$$w_1^\delta = w_1 + \delta \left(bt + |x|^2 \right).$$

Then, we have that w_1^δ satisfies

$$\partial_t w_1^\delta = \mu_1 \Delta w_1^\delta - \chi_1 (A_1 \nabla v) \cdot \nabla w_1^\delta - c w_1^\delta \\ + \delta \left(b + c(x,t) bt + c(x,t) |x|^2 + 2\chi_1 (A_1 \nabla v) \cdot x - 4\mu_1 \right).$$

We choose $b > 0$ such that

$$b \geq \sup_{x \in \mathbb{R}^2} \left\{ 4\mu_1 + 2\chi_1 |x| \, \||\nabla v|\|_{L^\infty([t_0,T)\times\mathbb{R}^2)} - |x|^2 \right\},$$

and conclude that

$$\delta \left(b + c(x,t) bt + c(x,t) |x|^2 + 2\chi_1 (A_1 \nabla v^\varepsilon) \cdot x - 4\mu_1 \right) \geq 0 \text{ in } \mathbb{R}^2 \times (t_0, t_1).$$

Now, we fix such an b and take $\delta > 0$ so small that

$$w_1^\delta(x_1, t_1) = w_1(x_1, t_1) + \delta \left(bt_1 + |x_1|^2 \right) \leq -a/2 < 0.$$

Since $u_1 \in BC([t_0, T) \times \mathbb{R}^2)$, we also have that $w_1 \in BC([t_0, T) \times \mathbb{R}^2)$. Moreover, if

$$|x| > \delta^{-1/2} \left(- \inf_{\mathbb{R}^2 \times (t_0, T)} w_1 \right)^{1/2} =: R, \; x \in \mathbb{R}^2,$$

then $w_1^\delta(x,t) > 0$ $(t \in (t_0, T))$. Since the function w_1^δ is continuous in $\mathbb{R}^2 \times [t_0, t_1]$, w_1^δ has a minimum value on $\overline{B_R(0)} \times [t_0, t_1]$. So, there exists a point $(x_3, t_3) \in \overline{B_R(0)} \times [t_0, t_1]$ such that

$$w_1^\delta(x_3, t_3) = \inf_{\overline{B_R(0)} \times [t_0, t_1]} w_1^\delta(x,t) \leq w_1^\delta(x_1, t_1) < 0.$$

So, since $w_1^\delta(x, t_0) \geq 0$ $(x \in \mathbb{R}^2)$ and $w_1^\delta(x,t) \geq 0$ $(|x| = R, t \in [t_0, t_1])$, we have that w_1^δ attains its minimum at a point $(x_3, t_3) \in B_R(0) \times (t_0, t_1]$, then

w_1^δ is a negative constant on $B_R(0) \times (t_0, t_3]$ but this is not possible due to the continuity of w_1^δ in $\mathbb{R}^2 \times [t_0, t_1]$. Therefore, $u_1(x,t) \geq 0$ on $(t_0, T) \times \mathbb{R}^2$. Moreover, if there is a point (x_4, t_4) in $\mathbb{R}^2 \times (t_0, T)$ at which $u_1(x_4, t_4) = 0$, then $w_1(x_4, t_4) = 0$ and $w_1 = 0$ in $B_r(0) \times (t_0, t_4]$, for all $r > |x_4|$, which implies that $w_1 = 0$ in $\mathbb{R}^2 \times (t_0, t_4]$ and this contradicts (3.46). So, we conclude that $u_1(x,t) > 0$ on $(t_0, T) \times \mathbb{R}^2$. and hence the positivity of u_1 on $(0,T) \times \mathbb{R}^2$ follows because t_0 is an arbitrary number in the interval $(0, T_{\delta^*})$. Similarly, we also obtain the positivity of u_2 on $(0,T) \times \mathbb{R}^2$. ■

Remark 12 *Notice that*

$$\begin{aligned}\int_0^t (t-s)^{x-1} s^{y-1} ds &= t \int_0^1 (t-ts)^{x-1} (ts)^{y-1} ds \\ &= t^{x+y-1} \int_0^1 (1-s)^{x-1} s^{y-1} ds \\ &= t^{x+y-1} \beta(x,y),\end{aligned} \tag{3.47}$$

where the function β is called the Beta function, and is defined by $\beta(x,y) := \int_0^1 (1-s)^{x-1} s^{y-1} ds$, which converges when $x, y > 0$. We observe

$$\begin{aligned}&\int_0^1 (1-s)^{x-1} s^{y-1} ds \\ &= \int_0^{1/2} (1-s)^{x-1} s^{y-1} ds + \int_{1/2}^1 (1-s)^{x-1} s^{y-1} ds \\ &\leq \int_0^{1/2} s^{y-1} ds + \int_{1/2}^1 (1-s)^{x-1} ds = \frac{1}{2^y y} + \frac{1}{2^x x} < \infty.\end{aligned}$$

3.2 Global existence

Our purpose in this section is to prove the two parts of theorem 2.

3.2.1 Case $\alpha_1, \alpha_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Theorem 13 *Assume that a_1, a_2 are non-negative constants and $\alpha_1, \alpha_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, i.e., both species move toward the gradient of chemical concentration. Let us denote by θ_i with $i = 1, 2$ the total initial masses define by (3.4). If θ_1 and θ_2 satisfy the inequalities*

$$\begin{aligned}\theta_1 < \frac{8\pi\mu_1}{\chi_1 a_1 \cos\alpha_1}, \ \theta_2 < \frac{8\pi\mu_2}{\chi_2 a_2 \cos\alpha_2}, \\ \text{and } \frac{8\pi\mu_1 a_1}{\chi_1 \cos\alpha_1}\theta_1 + \frac{8\pi\mu_2 a_2}{\chi_2 \cos\alpha_2}\theta_2 - (a_1\theta_1 + a_2\theta_2)^2 > 0,\end{aligned} \tag{3.48}$$

then system (3.1) has a global weak solution satisfying the energy dissipation (3.9) under the additional hypothesis $u_{10} |x|^2, u_{20} |x|^2 \in L^1(R^2)$.

We decompose the proof of the theorem 13 into three parts. In the first part, we constructed a regularized version of the system (3.1) having smooth solutions and introduced some of its properties like mass conservation, integrability, and positivity. Then, we showed in the second part how to obtain uniform estimates of the regularized system to pass to the limit, and obtained the result of global existence of weak solutions for the system (3.1). In the third part, we showed that the weak solutions of the system (3.1) satisfy the free-energy inequality (3.9).

Regularized Problem and some important properties Due to the unboundedness of the convolution kernel $\mathbf{K}(x) := -\frac{1}{2\pi}\ln|x|$ and its singularity at zero, we need to introduce a regularized version of our system.

We consider the regularized problem for $0 < \varepsilon < 1/\sqrt{2}$

$$\begin{array}{ll} \partial_t u_1^\varepsilon = \mu_1 \triangle u_1^\varepsilon - \chi_1 \nabla \cdot (u_1^\varepsilon A_1 \nabla v^\varepsilon), & x \in \mathbb{R}^2, t > 0, \\ \partial_t u_2^\varepsilon = \mu_2 \triangle u_2^\varepsilon - \chi_2 \nabla \cdot (u_2^\varepsilon A_2 \nabla v^\varepsilon), & x \in \mathbb{R}^2, t > 0, \\ \nabla v^\varepsilon := \nabla K^\varepsilon * (a_1 u_1^\varepsilon + a_2 u_2^\varepsilon), & x \in \mathbb{R}^2, t > 0, \\ u_1^\varepsilon(x,0) = u_{10}(x), u_2^\varepsilon(x,0) = u_{20}(x), & x \in \mathbb{R}^2. \end{array} \tag{3.49}$$

Here $\mathbf{K}^\varepsilon$ is defined as

$$\mathbf{K}^\varepsilon(x) := \frac{1}{4\pi}\ln\frac{1}{|x|^2+\varepsilon^2}. \tag{3.50}$$

Simple computations show that

$$\nabla\mathbf{K}^\varepsilon(x) = -\frac{1}{2\pi}\frac{x}{|x|^2+\varepsilon^2}, \quad \triangle\mathbf{K}^\varepsilon(x) = -\frac{1}{\pi}\frac{\varepsilon^2}{\left(|x|^2+\varepsilon^2\right)^2}.$$

Notice that

$$|\nabla\mathbf{K}^\varepsilon(x)| \leq \frac{1}{4\pi\varepsilon},, \quad |\nabla\mathbf{K}^\varepsilon(x)| \leq \frac{1}{2\pi}\frac{1}{|x|},$$

$$\text{and } \|\triangle\mathbf{K}^\varepsilon\|_{L^1} = 2\varepsilon^2\int_0^\infty \frac{r}{\left(r^2+\varepsilon^2\right)^2}dr = 1.$$

This regularization has been already used for different kinds of Keller-Segel-type models (e.g. [19, 20]). The advantage of this regularized version subject to initial conditions satisfying (3.6) is that it possesses a unique positive global smooth solution with fast decay in space $(u_1^\varepsilon, u_2^\varepsilon) \in BC\left([0,T);L^1\left(\mathbb{R}^2\right)\right) \cap C^{2,1}\left(\mathbb{R}^2\times(0,T)\right) \cap X_T$, where X_T is the Banach space defined by (3.22) with norm 3.23. Moreover, for any $1 \leq p \leq \infty$, there holds $u_i^\varepsilon \in L^\infty((0,T);L^p(\mathbb{R}^2))$, $i = 1,2$. In particular, the masses $\int_{\mathbb{R}^2} u_1^\varepsilon(\cdot,t)dx$ and $\int_{\mathbb{R}^2} u_2(\cdot,t)dx$ remain constants in time. This result can be proved by adapting the techniques in [64], as illustrated in the following proposition.

Proposition 14 *Assume that u_{10}, u_{20} satisfy (3.6) and (3.4). Then there is a unique classic solution $(u_1^\varepsilon, u_2^\varepsilon) \in BC\left([0,T);L^1\left(\mathbb{R}^2\right)\right) \cap C^{2,1}\left(\mathbb{R}^2\times(0,T)\right) \cap X_T$, of (3.49) with $0 < \varepsilon < 1/\sqrt{2}$ on $[0,T)$, for any $0 < T < \infty$. Then $(u_1^\varepsilon, u_2^\varepsilon)$ satisfies the following properties:*

(i) *mass conservation, i.e., $\int_{\mathbb{R}^2} u_i^\varepsilon(x,t)dx = \theta_i$, for $i = 1,2$ and $t \in [0,T)$;*

(ii) *integrability, i.e., for every* $1 \le p \le \infty$, *there holds* $u_i^\varepsilon \in L^\infty((0,T);L^p(\mathbb{R}^2))$ *for* $i = 1,2$;

(iii) positivity, i.e., $u_i^\varepsilon(x,t) > 0$ for all $(x,t) \in \mathbb{R}^2 \times (0,T)$ for $i = 1,2$;

(iv) $u_i^\varepsilon \ln(1+|x|^2) \in L^\infty((0,T);L^1(\mathbb{R}^2))$ *for* $i = 1,2$;

(v) $u_i^\varepsilon \ln u_i^\varepsilon \in L^\infty((0,T);L^1(\mathbb{R}^2))$ *for* $i = 1,2$;

(vi) $|\nabla(\sqrt{u_i^\varepsilon})| \in L^2((0,T);L^2(\mathbb{R}^2))$ *for* $i = 1,2$;

(vii) $u_i^\varepsilon v^\varepsilon \in L^\infty((0,T);L^1(\mathbb{R}^2))$ *for* $i = 1,2$;

(viii) for every $2 < p < \infty$, there holds $\nabla v^\varepsilon \in L^\infty((0,T);W^{1,p}(\mathbb{R}^2))^2$.

Proof. The *local existence* and *mass conservation* results of a unique classical solution $(u_1^\varepsilon, u_2^\varepsilon) \in BC([0,T);L^1(\mathbb{R}^2)) \cap C^{2,1}(\mathbb{R}^2 \times (0,T)) \cap X_T$ of (3.49) with $\varepsilon > 0$ can be proved by the same argument of Proposition 11 with minor modifications.

(ii) *Integrability.* Notice that the smoothness of the solution $(u_1^\varepsilon, u_2^\varepsilon)$ allows us to apply the Duhamel integral formula to obtain

$$u_i^\varepsilon(t) = e^{\mu_i t\Delta} u_{i0} - \chi_i \int_0^t e^{\mu_i (t-s)\Delta} \nabla \cdot (u_i^\varepsilon(s) A_i \nabla v^\varepsilon(s))\, ds. \tag{3.51}$$

Using (3.18), (3.19), we get for any $1 \le q \le p \le \infty$ that

$$\begin{aligned}
&\|u_i^\varepsilon\|_{L^p(\mathbb{R}^2)} \\
&\le \left\|e^{\mu_i t\Delta} u_{i0}\right\|_{L^p(\mathbb{R}^2)} + \chi_i \int_0^t \left\|e^{\mu_i (t-s)\Delta} \nabla \cdot (u_i^\varepsilon(s) A_i \nabla v^\varepsilon(s))\right\|_{L^p(\mathbb{R}^2)} ds \\
&\le \|u_{i0}\|_{L^p(\mathbb{R}^2)} + C_1 \chi_i \int_0^t (t-s)^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}} \|u_i^\varepsilon(s) A_i \nabla v^\varepsilon(s)\|_{L^q(\mathbb{R}^2)} ds.
\end{aligned} \tag{3.52}$$

For $1 < p < 2$, taking $q = 1$ so that $-\frac{1}{2} + \frac{1}{p} - \frac{1}{q} + 1 = \frac{1}{p} - \frac{1}{2} > 0$, (3.52) becomes

$$\begin{aligned}
&\|u_i^\varepsilon\|_{L^p(\mathbb{R}^2)} \\
&\le \|u_{i0}\|_{L^p(\mathbb{R}^2)} + C_1 \chi_i \int_0^t (t-s)^{\frac{1}{p}-\frac{3}{2}} \|u_i^\varepsilon(s) A_i \nabla v^\varepsilon(s)\|_{L^1(\mathbb{R}^2)} ds \\
&\le \|u_{i0}\|_{L^p(\mathbb{R}^2)} + C_1 \chi_i \, \||\nabla \mathbf{K}^\varepsilon|\|_{L^\infty(\mathbb{R}^2)} \\
&\int_0^t (t-s)^{\frac{1}{p}-\frac{3}{2}} \|u_i^\varepsilon(s)\|_{L^1(\mathbb{R}^2)} \|(a_1 u_1^\varepsilon + a_2 u_2^\varepsilon)(s)\|_{L^1(\mathbb{R}^2)} ds \\
&\le \|u_{i0}\|_{L^p(\mathbb{R}^2)} + \frac{\max\{|a_1|, |a_2|\} \chi_i C_1}{4\pi\varepsilon} \int_0^t (t-s)^{\frac{1}{p}-\frac{3}{2}} ds \\
&\left(\|u_1^\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}^2))} + \|u_2^\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}^2))}\right)^2 \\
&= \|u_{i0}\|_{L^p(\mathbb{R}^2)} + \frac{p \max\{|a_1|, |a_2|\} \chi_i C_1}{2\pi(2-p)\varepsilon} \\
&\left(\|u_1^\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}^2))} + \|u_2^\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}^2))}\right)^2 t^{\frac{1}{p}-\frac{1}{2}}.
\end{aligned}$$

For $p = 2$, taking $q = 4/3 < 2$, (3.52) becomes

$$\begin{aligned}
&\|u_i^\varepsilon\|_{L^2(\mathbb{R}^2)} \\
&\le \|u_{i0}\|_{L^2(\mathbb{R}^2)} + C_1\chi_i \int_0^t (t-s)^{-\frac{3}{4}} \|u_i^\varepsilon(s) A_i \nabla v^\varepsilon(s)\|_{L^{4/3}(\mathbb{R}^2)} ds \\
&\le \|u_{i0}\|_{L^2(\mathbb{R}^2)} + \frac{\max\{|a_1|, |a_2|\}\chi_i C_1}{4\pi\varepsilon} \int_0^t (t-s)^{-\frac{3}{4}} ds \\
&\|u_i^\varepsilon\|_{L^\infty((0,T);L^{4/3}(\mathbb{R}^2))} \left(\|u_1^\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}^2))} + \|u_2^\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}^2))}\right) \\
&= \|u_{i0}\|_{L^2(\mathbb{R}^2)} + \frac{\max\{|a_1|, |a_2|\}\chi_i C_1}{\pi\varepsilon} \|u_i^\varepsilon\|_{L^\infty((0,T);L^{4/3}(\mathbb{R}^2))} \\
&\left(\|u_1^\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}^2))} + \|u_2^\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}^2))}\right) t^{1/4}.
\end{aligned}$$

For $2 < p < \infty$,taking $q = 2$ so that $-\frac{1}{2} + \frac{1}{p} - \frac{1}{q} + 1 = \frac{1}{p} > 0$, (3.52) becomes

$$\begin{aligned}
&\|u_i^\varepsilon\|_{L^p(\mathbb{R}^2)} \\
&\le \|u_{i0}\|_{L^p(\mathbb{R}^2)} + C_1\chi_i \int_0^t (t-s)^{\frac{1}{p}-1} \|u_i^\varepsilon(s) A_i \nabla v^\varepsilon(s)\|_{L^2(\mathbb{R}^2)} ds \\
&\le \|u_{i0}\|_{L^p(\mathbb{R}^2)} + \frac{\max\{|a_1|, |a_2|\}\chi_i C_1}{4\pi\varepsilon} \int_0^t (t-s)^{\frac{1}{p}-1} ds \\
&\|u_i^\varepsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^2))} \left(\|u_1^\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}^2))} + \|u_2^\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}^2))}\right) \\
&= \|u_{i0}\|_{L^p(\mathbb{R}^2)} + \frac{p\max\{|a_1|, |a_2|\}\chi_i C_1}{4\pi\varepsilon} \|u_i^\varepsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^2))} \\
&\left(\|u_1^\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}^2))} + \|u_2^\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}^2))}\right) t^{1/p}.
\end{aligned}$$

For $p = +\infty$, taking $q = 4$ so that $-\frac{1}{2} + \frac{1}{p} - \frac{1}{q} + 1 = \frac{1}{4} > 0$, (3.52) becomes

$$\begin{aligned}
&\|u_i^\varepsilon\|_{L^\infty(\mathbb{R}^2)} \\
&\le \|u_{i0}\|_{L^\infty(\mathbb{R}^2)} + C_1\chi_i \int_0^t (t-s)^{-\frac{3}{4}} \|u_i^\varepsilon(s) A_i \nabla v^\varepsilon(s)\|_{L^4(\mathbb{R}^2)} ds \\
&\le \|u_{i0}\|_{L^\infty(\mathbb{R}^2)} + \frac{\max\{|a_1|, |a_2|\}\chi_i C_1}{4\pi\varepsilon} \int_0^t (t-s)^{-\frac{3}{4}} ds \\
&\|u_i^\varepsilon\|_{L^\infty((0,T);L^4(\mathbb{R}^2))} \left(\|u_1^\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}^2))} + \|u_2^\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}^2))}\right) \\
&= \|u_{i0}\|_{L^\infty(\mathbb{R}^2)} + \frac{\max\{|a_1|, |a_2|\}\chi_i C_1}{\pi\varepsilon} \|u_i^\varepsilon\|_{L^\infty((0,T);L^4(\mathbb{R}^2))} \\
&\left(\|u_1^\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}^2))} + \|u_2^\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}^2))}\right) t^{1/4}.
\end{aligned}$$

(iii) Multiplying the first equation of system (3.49) by $(u_1^\varepsilon)^- := \max\{0, -u_1^\varepsilon\}$, integrating over $\mathbb{R}^2$ and integrating by parts

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\left|(u_1^\varepsilon)^-\right|^2dx\\
&=-\mu_1\int_{\mathbb{R}^2}\left|\nabla(u_1^\varepsilon)^-\right|^2dx+\chi_1\int_{\mathbb{R}^2}\nabla(u_1^\varepsilon)^-\cdot(u_1^\varepsilon A_1\nabla v^\varepsilon)dx\\
&=-\mu_1\int_{\mathbb{R}^2}\left|\nabla(u_1^\varepsilon)^-\right|^2dx+\frac{\chi_1}{2}\int_{\mathbb{R}^2}\nabla\left|(u_1^\varepsilon)^-\right|^2\cdot(A_1\nabla v^\varepsilon)dx\\
&=-\mu_1\int_{\mathbb{R}^2}\left|\nabla(u_1^\varepsilon)^-\right|^2dx-\frac{\chi_1}{2}\int_{\mathbb{R}^2}\left|(u_1^\varepsilon)^-\right|^2\nabla\cdot(A_1\nabla v^\varepsilon)dx.
\end{aligned}$$

Using the identity $\nabla\cdot A_1\nabla v^\varepsilon=\cos\alpha_1\Delta v^\varepsilon$,we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\left|(u_1^\varepsilon)^-\right|^2dx\\
&=-\mu_1\int_{\mathbb{R}^2}\left|\nabla(u_1^\varepsilon)^-\right|^2dx+\frac{\chi_1\cos\alpha_1}{2}\int_{\mathbb{R}^2}\left|(u_1^\varepsilon)^-\right|^2(-\Delta v^\varepsilon)dx.
\end{aligned}$$

Note that by Hölder's inequality and Young's inequality for convolutions, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left|(u_1^\varepsilon)^-\right|^2\left|\left(-\Delta\mathbf{K}^\varepsilon*(a_1u_1^\varepsilon+a_2u_2^\varepsilon)\right)(s)\right|dx\\
&\le\left\|\left|(u_1^\varepsilon)^-\right|^2\right\|_{L^2(\mathbb{R}^2)}\left\||-\Delta\mathbf{K}^\varepsilon*(a_1u_1^\varepsilon+a_2u_2^\varepsilon)|\right\|_{L^2(\mathbb{R}^2)}\\
&\le\left\|\left|(u_1^\varepsilon)^-\right|^2\right\|_{L^2(\mathbb{R}^2)}\left\|\Delta\mathbf{K}^\varepsilon\right\|_{L^1(\mathbb{R}^2)}\left\|a_1u_1^\varepsilon+a_2u_2^\varepsilon\right\|_{L^2(\mathbb{R}^2)}\\
&=\left\|\left|(u_1^\varepsilon)^-\right|\right\|_{L^4(\mathbb{R}^2)}^2\left\|a_1u_1^\varepsilon+a_2u_2^\varepsilon\right\|_{L^2(\mathbb{R}^2)}. \qquad (3.53)
\end{aligned}$$

At this stage, we use item (**ii**) to claim that $u_i^\varepsilon\in L^\infty((0,T);L^2(\mathbb{R}^2))$, for $i=1,2$,and

$$\|u_1^\varepsilon(t)\|_{L^\infty((0,T);L^2(\mathbb{R}^2))}+\|u_2^\varepsilon(t)\|_{L^\infty((0,T);L^2(\mathbb{R}^2))}\le C_2,$$

for some constant $C_2=C(\varepsilon)$ together with (3.53) to derive

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}\left|(u_1^\varepsilon)^-\right|^2dx \qquad (3.54)\\
&\le-2\mu_1\int_{\mathbb{R}^2}\left|\nabla(u_1^\varepsilon)^-\right|^2dx+\chi_1\cos\alpha_1\max\{a_1,a_2\}C_2\left\|\left|(u_1^\varepsilon)^-\right|\right\|_{L^4(\mathbb{R}^2)}^2.
\end{aligned}$$

Notice that $(u_1^\varepsilon)^-(\cdot,t)\in H^1(\mathbb{R}^2)$ since $u_1^\varepsilon(\cdot,t)\in H^1(\mathbb{R}^2)$ (See [40, Theorem 4 p. 130]). By the interpolation inequality of Gagliardo-Nirenberg-Sobolev (3.45), we get

$$\left\|(u_1^\varepsilon)^-\right\|_{L^4(\mathbb{R}^2)}^2\le C_3\left\|(u_1^\varepsilon)^-\right\|_{L^2(\mathbb{R}^2)}\left\|\nabla(u_1^\varepsilon)^-\right\|_{L^2(\mathbb{R}^2)}. \qquad (3.55)$$

It follows from estimates (3.54) and (3.55)

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}\left|(u_1^\varepsilon)^-\right|^2 dx \leq -2\mu_1\int_{\mathbb{R}^2}\left|\nabla(u_1^\varepsilon)^-\right|^2 dx\\
&+\frac{C_4}{\sqrt{2\mu_1}}\left(\int_{\mathbb{R}^2}\left|(u_1^\varepsilon)^-\right|^2 dx\right)^{1/2}\left(2\mu_1\int_{\mathbb{R}^2}\left|\nabla(u_1^\varepsilon)^-\right|^2 dx\right)^{1/2}\\
&\leq -\mu_1\int_{\mathbb{R}^2}\left|\nabla(u_1^\varepsilon)^-\right|^2 dx+\frac{(C_4)^2}{4\mu_1}\left(\int_{\mathbb{R}^2}\left|(u_1^\varepsilon)^-\right|^2 dx\right)\\
&\leq C_5\left(\int_{\mathbb{R}^2}\left|(u_1^\varepsilon)^-\right|^2 dx\right).
\end{aligned}$$

Integrating the last inequality

$$\int_{\mathbb{R}^2}\left|(u_1^\varepsilon)^-\right|^2 dx \leq \left(\int_{\mathbb{R}^2}\left|(u_1^\varepsilon(x,0))^-\right|^2 dx\right)e^{C_5 t}=0,$$

and therefore $(u_1^\varepsilon)^-=0$ on $[0,T)\times\mathbb{R}^2$. Hence, $u_1^\varepsilon\geq 0$ on $[0,T)\times\mathbb{R}^2$. From

$$\int_{\mathbb{R}^2} u_1^\varepsilon(x,t)dx=\theta_1>0,\ \text{for all } t\in[0,T)\,, \tag{3.56}$$

we have that $u_1^\varepsilon(\cdot,t)\not\equiv 0, u_1^\varepsilon(x,t)\geq 0$ for any $t\in[0,T)$. Similarly, we also obtain that $u_2^\varepsilon(\cdot,t)\not\equiv 0, u_2^\varepsilon(x,t)\geq 0$ for any $t\in[0,T)$.

Now, we proceed to show that non-negativity of u_1^ε and u_2^ε together with the fact that $(u_1^\varepsilon,u_2^\varepsilon)\in BC\left([0,T)\times\mathbb{R}^2\right)^2$, for $i=1,2$, is a classical solution of

$$\begin{aligned}
\partial_t u_1^\varepsilon&=\mu_1\Delta u_1^\varepsilon-\chi_1(A_1\nabla v^\varepsilon)\cdot\nabla u_1^\varepsilon-\chi_1\cos\alpha_1\Delta v^\varepsilon u_1^\varepsilon,\quad x\in\mathbb{R}^2,t\in(0,T)\,,\\
\partial_t u_2^\varepsilon&=\mu_2\Delta u_2^\varepsilon-\chi_2(A_2\nabla v^\varepsilon)\cdot\nabla u_2^\varepsilon-\chi_2\cos\alpha_2\Delta v^\varepsilon u_2^\varepsilon,\quad x\in\mathbb{R}^2,t\in(0,T)\,,
\end{aligned}$$

and the strong maximum principle (See [39, Theorem 12 p. 376]) implies that $u_i^\varepsilon(x,t)>0, i=1,2,$ on $(0,T)\times\mathbb{R}^2$. With this end in mind, we start by defining $w_1^\varepsilon=u_1^\varepsilon e^{-\lambda t}$, with $\lambda:=\chi_1\cos\alpha_1\left\|\Delta v^\varepsilon\right\|_{L^\infty([0,T)\times\mathbb{R}^2)}$. Then w_1^ε satisfies

$$\partial_t w_1^\varepsilon=\mu_1\Delta w_1^\varepsilon-\chi_1(A_1\nabla v^\varepsilon)\cdot\nabla w_1^\varepsilon-c(x,t)w_1^\varepsilon,\quad x\in\mathbb{R}^2,t\in(0,T)\,,$$

and

$$\begin{aligned}
c(x,t)&:=\chi_1\cos\alpha_1\Delta v^\varepsilon(x,t)+\lambda\\
&=\chi_1\cos\alpha_1\left(\Delta v^\varepsilon(x,t)+\left\|\Delta v^\varepsilon\right\|_{L^\infty([0,T)\times\mathbb{R}^2)}\right)\geq 0.
\end{aligned}$$

Moreover, notice that $\left|\nabla v^\varepsilon\right|,\Delta v^\varepsilon\in BC\left([0,T)\times\mathbb{R}^2\right)$[4]. We assume by contradiction $u_1^\varepsilon(x_1,t_1)=0$ at $(x_1,t_1)\in\mathbb{R}^2\times(t_0,T)$, then $w_1^\varepsilon(x_1,t_1)=0$ and by the strong maximum principle $w_1^\varepsilon=0$ in $B_r(0)\times(0,t_1]$, for all $r>|x_1|$, which implies that $w_1^\varepsilon=0$ in $\mathbb{R}^2\times(0,t_1]$ and this contradicts (3.56). So, we conclude that $u_1^\varepsilon(x,t)>0$ on $(0,T)\times\mathbb{R}^2$. Similarly, we also obtain the positivity of u_2^ε on $(0,T)\times\mathbb{R}^2$.

---

[4]Notice that
$\left\|\left|\nabla v^\varepsilon(t)\right|\right\|_{L^\infty(\mathbb{R}^2)}\leq C\left\|a_1u_1^\varepsilon(t)+a_2u_2^\varepsilon(t)\right\|_{L^1(\mathbb{R}^2)}^{1/4}\left\|a_1u_1^\varepsilon(t)+a_2u_2^\varepsilon(t)\right\|_{L^3(\mathbb{R}^2)}^{3/4},$
and
$\left\|\Delta v^\varepsilon(t)\right\|_{L^\infty(\mathbb{R}^2)}\leq\left\|\left(a_1u_1^\varepsilon+a_2u_2^\varepsilon\right)(t)\right\|_{L^\infty(\mathbb{R}^2)}.$

To establish *global existence*, we employ a proof by contradiction (See [85]): Let $T_{\max}$ be the maximal existence time of the mild solution $(u_1^\varepsilon, u_2^\varepsilon)$ of the (3.49) with initial data (u_{10}, u_{20}) and $T_{\max} < +\infty$, (3.51) implies

$$e^{\mu_i \tau \Delta} u_i^\varepsilon(t) = e^{\mu_i(\tau+t)\Delta} u_{i0} - \chi_i \int_0^t e^{\mu_i(\tau+t-s)\Delta} \nabla \cdot (u_i^\varepsilon(s) A_i \nabla v^\varepsilon(s))\, ds,$$

for all $\tau > 0, t \in (0, T_{\max})$. We claim that

$$\sup_{\tau \in (0, 2(T_{\max}-t))} \tau^{1/4} \left\| e^{\mu_i \tau \Delta} u_i^\varepsilon(t) \right\|_{L^{4/3}(\mathbb{R}^2)} \to 0 \text{ as } t \to T_{\max}. \tag{3.57}$$

Indeed, by (3.18) we have that

$$\begin{aligned} &\sup_{\tau \in (0, 2(T_{\max}-t))} \tau^{1/4} \left\| e^{\mu_i(\tau+t)\Delta} u_{i0} \right\|_{L^{4/3}(\mathbb{R}^2)} \\ &\le \sup_{\tau \in (0, 2(T_{\max}-t))} \left(\frac{\tau}{\tau+t} \right)^{1/4} (4\pi\mu_i)^{-1/4} \left\| u_{i0} \right\|_{L^1(\mathbb{R}^2)} \\ &= \left(\frac{2(T_{\max}-t)}{2T_{\max}-t} \right)^{1/4} (4\pi\mu_i)^{-1/4} \theta_i \to 0 \text{ as } t \to T_{\max}. \end{aligned}$$

On the other hand, by (3.19) we get

$$\begin{aligned} &\int_0^t \left\| e^{\mu_i(\tau+t-s)\Delta} \nabla \cdot (u_i^\varepsilon(s) A_i \nabla v^\varepsilon(s)) \right\|_{L^{4/3}(\mathbb{R}^2)} ds \\ &\le C_6 \int_0^t (\tau+t-s)^{-\frac{1}{2}+\frac{3}{4}-1} \left\| |u_i^\varepsilon(s) A_i \left(\nabla \mathbf{K}^\varepsilon * (a_1 u_1^\varepsilon + a_2 u_2^\varepsilon) \right)(s)| \right\|_{L^1(\mathbb{R}^2)} ds \\ &\le C_6 \int_0^t (\tau+t-s)^{-\frac{3}{4}} \theta_i \left\| |\nabla \mathbf{K}^\varepsilon * (a_1 u_1^\varepsilon + a_2 u_2^\varepsilon)(s)| \right\|_{L^\infty(\mathbb{R}^2)} ds \\ &\le \max\{|a_1|, |a_2|\} (\theta_1+\theta_2)^2 \left\| |\nabla \mathbf{K}^\varepsilon| \right\|_{L^\infty(\mathbb{R}^2)} C_6 \int_0^t (\tau+t-s)^{-\frac{3}{4}} ds \\ &\le \frac{\max\{|a_1|, |a_2|\} (\theta_1+\theta_2)^2 C_6}{4\pi\varepsilon} \int_0^t (\tau+t-s)^{-\frac{3}{4}} ds \\ &= \frac{\max\{|a_1|, |a_2|\} (\theta_1+\theta_2)^2 C_6}{\pi\varepsilon} \left((\tau+t)^{1/4} - \tau^{1/4} \right) \\ &\le \frac{\max\{|a_1|, |a_2|\} (\theta_1+\theta_2)^2 C_6}{\pi\varepsilon} (\tau+t)^{1/4}. \end{aligned}$$

Then

$$\begin{aligned} &\sup_{\tau \in (0, 2(T_{\max}-t))} \tau^{1/4} \left\| \int_0^t e^{\mu_i(\tau+t-s)\Delta} \nabla \cdot (u_i^\varepsilon(s) A_i \nabla v^\varepsilon(s))\, ds \right\|_{L^{4/3}(\mathbb{R}^2} \\ &\le \frac{\max\{|a_1|, |a_2|\} (\theta_1+\theta_2)^2 C_6}{\pi\varepsilon} \sup_{\tau \in (0, 2(T_{\max}-t))} \tau^{1/4} (\tau+t)^{1/4} \\ &= \frac{\max\{|a_1|, |a_2|\} (\theta_1+\theta_2)^2 C_6}{\pi\varepsilon} (2(T_{\max}-t))^{1/4} (2T_{\max}-t)^{1/4} \to 0, \end{aligned}$$

as $t \to T_{\max}$. However, (3.57) contradicts $T_{\max} < +\infty$. In fact, for $0 < \delta < 1/(4C_B)$, there exists $t_0 \in (0, T_{\max})$ such that

$$\sup_{0<\tau<2(T_{\max}-t_0)} \tau^{1/4} \left(\left\| e^{\mu_1 \tau \Delta} u_1^\varepsilon(t_0) \right\|_{L^{4/3}(\mathbb{R}^2)} + \left\| e^{\mu_2 \tau \Delta} u_2^\varepsilon(t_0) \right\|_{L^{4/3}(\mathbb{R}^2)} \right) < \delta.$$

By doing so, we can extend the solution to the time $2T_{\max} - t_0$ starting from the time t_0. Thus $T_{\max} = +\infty$.

(**iv**) Note that

$$\left|\nabla \ln(1+|x|^2)\right| = \frac{2|x|}{1+|x|^2} \le 1, \text{ and } \left|\Delta \ln(1+|x|^2)\right| = \frac{4}{\left(1+|x|^2\right)^2} \le 4.$$

Multiplying the first equation of system (3.49) by $\ln(1+|x|^2)$ and integrating over $\mathbb{R}^2$, we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} u_1^\varepsilon \ln(1+|x|^2)dx \\
&= \mu_1 \int_{\mathbb{R}^2} u_1^\varepsilon \Delta \ln(1+|x|^2)dx + \chi_1 \int_{\mathbb{R}^2} \nabla \ln(1+|x|^2)\cdot(u_1^\varepsilon A_1 \nabla v^\varepsilon)dx \\
&\le 4\mu_1\theta_1 + \chi_1 \int_{\mathbb{R}^2} u_1^\varepsilon \left|\nabla v^\varepsilon\right| dx \\
&\le 4\mu_1\theta_1 + \chi_1 \left\|u_1^\varepsilon(t)\right\|_{L^{4/3}(\mathbb{R}^2)} \left\|\left|\nabla v^\varepsilon(t)\right|\right\|_{L^4(\mathbb{R}^2)}.
\end{aligned}$$

Using (3.20), we have that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} u_1^\varepsilon \ln(1+|x|^2)dx \le 4\mu_1\theta_1 \\
&+ \frac{\max\{|a_1|,|a_2|\}C_7\chi_1}{2\pi}\left(\|u_1^\varepsilon\|_{L^\infty((0,T);L^{4/3}(\mathbb{R}^2))} + \|u_2^\varepsilon\|_{L^\infty((0,T);L^{4/3}(\mathbb{R}^2))}\right)^2.
\end{aligned}$$

By **(ii)**, we have that

$$\frac{d}{dt}\int_{\mathbb{R}^2} u_1^\varepsilon \ln(1+|x|^2)dx \le C_8(\varepsilon).$$

Integrating on the interval $(0,t)$, we obtain that

$$\int_{\mathbb{R}^2} u_1^\varepsilon \ln(1+|x|^2)dx \le \int_{\mathbb{R}^2} u_{10} \ln(1+|x|^2)dx + C_8(\varepsilon)T, \tag{3.58}$$

for any $t \in [0,T]$. Similarly,

$$\int_{\mathbb{R}^2} u_2^\varepsilon \ln(1+|x|^2)dx \le \int_{\mathbb{R}^2} u_{20} \ln(1+|x|^2)dx + C_9(\varepsilon)T, \tag{3.59}$$

for any $t \in [0,T]$.

(**v**) Notice that

$$\begin{aligned}
-1_{\{u_i^\varepsilon \le 1\}} u_i^\varepsilon \ln u_i^\varepsilon &= 1_{\left\{u_i^\varepsilon \le \left(1+|x|^2\right)^{-4}\right\}} u_i^\varepsilon \ln\frac{1}{u_i^\varepsilon} + 1_{\left\{\left(1+|x|^2\right)^{-4} \le u_i^\varepsilon \le 1\right\}} u_i^\varepsilon \ln\frac{1}{u_i^\varepsilon} \\
&= 1_{\left\{u_i^\varepsilon \le \left(1+|x|^2\right)^{-4}\right\}} (u_i^\varepsilon)^{\frac{1}{2}} + 1_{\left\{\left(1+|x|^2\right)^{-4} \le u_i^\varepsilon \le 1\right\}} 4u_i^\varepsilon \ln(1+|x|^2) \\
&\le \left(1+|x|^2\right)^{-2} + 4u_i^\varepsilon \ln(1+|x|^2), i = 1,2.
\end{aligned} \tag{3.60}$$

Then,

$$\begin{aligned}
\int_{\mathbb{R}^2} u_i^\varepsilon \ln^- u_i^\varepsilon dx &\le \int_{\mathbb{R}^2} \left(1+|x|^2\right)^{-2} dx + 4\int_{\mathbb{R}^2} u_i^\varepsilon \ln(1+|x|^2)dx \\
&= \pi + 4\int_{\mathbb{R}^2} u_i^\varepsilon \ln(1+|x|^2)dx, i = 1,2.
\end{aligned} \tag{3.61}$$

Therefore

$$\int_{\mathbb{R}^2} u_i^\varepsilon \left|\ln u_i^\varepsilon\right| dx = \int_{\mathbb{R}^2} u_i^\varepsilon \ln^+ u_i^\varepsilon dx + \int_{\mathbb{R}^2} u_i^\varepsilon \ln^- u_i^\varepsilon dx$$
$$\leq \|u_i^\varepsilon\|^2_{L^\infty((0,T);L^2(\mathbb{R}^2))} + \pi + 4\int_{\mathbb{R}^2} u_i^\varepsilon \ln(1+|x|^2)dx, i = 1, 2.$$

(**vi**) Multiplying the first equation of system (3.49) by $\ln u_1^\varepsilon$, integrating over $\mathbb{R}^2$ and integrating by parts, we get

$$\frac{d}{dt}\int_{\mathbb{R}^2} u_1^\varepsilon \ln u_1^\varepsilon dx = -4\mu_1 \int_{\mathbb{R}^2} \left|\nabla\sqrt{u_1^\varepsilon}\right|^2 - \chi_1 \int_{\mathbb{R}^2} u_1^\varepsilon \nabla\cdot(A_1\nabla v^\varepsilon).$$

Using the identity $\nabla\cdot A_1\nabla v^\varepsilon = \cos\alpha_1 \Delta v^\varepsilon$, we deduce that

$$\frac{d}{dt}\int_{\mathbb{R}^2} u_1^\varepsilon \ln u_1^\varepsilon dx = -4\mu_1 \int_{\mathbb{R}^2} \left|\nabla\sqrt{u_1^\varepsilon}\right|^2 dx + \chi_1\cos\alpha_1 \int_{\mathbb{R}^2} u_1^\varepsilon\left(-\Delta v^\varepsilon\right) dx$$

By Hölder's inequality and Young's inequality for convolutions, we have that

$$\int_{\mathbb{R}^2} u_1^\varepsilon\left(-\Delta \mathbf{K}^\varepsilon * (a_1u_1^\varepsilon + a_2u_2^\varepsilon)\right) dx \leq \|u_1^\varepsilon\|_{L^2(\mathbb{R}^2)} \|a_1u_1^\varepsilon + a_2u_2^\varepsilon\|_{L^2(\mathbb{R}^2)}.$$

Then,

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2} u_1^\varepsilon \ln u_1^\varepsilon dx \\ &\leq -4\mu_1 \int_{\mathbb{R}^2} \left|\nabla\sqrt{u_1^\varepsilon}\right|^2 dx \\ &+ \chi_1 \max\{|a_1|, |a_2|\} \left|\cos\alpha_1\right| \left(\|u_1^\varepsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^2))} + \|u_2^\varepsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^2))}\right)^2.\end{aligned}$$

Integrating over $(0,T)$, we get

$$\begin{aligned}&4\mu_1 \int_0^T\int_{\mathbb{R}^2} \left|\nabla\sqrt{u_1^\varepsilon}\right|^2 dx \\ &\leq \chi_1 \max\{|a_1|, |a_2|\} \left|\cos\alpha_1\right| \left(\|u_1^\varepsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^2))} + \|u_2^\varepsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^2))}\right)^2 T \\ &+ \int_{\mathbb{R}^2} u_{10}\ln u_{10} dx - \int_{\mathbb{R}^2} u_1^\varepsilon(x,T)\ln u_1^\varepsilon(x,T) dx. \qquad (3.62)\end{aligned}$$

Similarly,

$$\begin{aligned}&4\mu_2 \int_0^T\int_{\mathbb{R}^2} \left|\nabla\sqrt{u_2^\varepsilon}\right|^2 dx \\ &\leq \chi_2 \max\{|a_1|, |a_2|\} \left|\cos\alpha_2\right| \left(\|u_1^\varepsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^2))} + \|u_2^\varepsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^2))}\right)^2 T \\ &+ \int_{\mathbb{R}^2} u_{20}\ln u_{20} dx - \int_{\mathbb{R}^2} u_2^\varepsilon(x,T)\ln u_2^\varepsilon(x,T) dx. \qquad (3.63)\end{aligned}$$

(**vii**) Using that $v^\varepsilon := \mathbf{K}^\varepsilon * (a_1u_1^\varepsilon + a_2u_2^\varepsilon)$, we have that

$$\begin{aligned}&\int_{\mathbb{R}^2} \left|u_i^\varepsilon v^\varepsilon\right| dx \\ &= \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \left|\ln\frac{1}{|x-y|^2+\varepsilon^2}\right| u_i^\varepsilon(x,t) \left|a_1u_1^\varepsilon(y,t) + a_2u_2^\varepsilon(y,t)\right| dydx.\end{aligned}$$

On the one hand, notice that

$$\begin{aligned}
&\int_{\mathbb{R}^2\times\mathbb{R}^2} \ln^+ \frac{1}{|x-y|^2+\varepsilon^2} u_i^\varepsilon(x,t)\,|a_1u_1^\varepsilon(y,t)+a_2u_2^\varepsilon(y,t)|\,dydx\\
&= \int_{|x-y|^2+\varepsilon^2\le 1} \ln \frac{1}{|x-y|^2+\varepsilon^2} u_i^\varepsilon(x,t)\,|a_1u_1^\varepsilon(y,t)+a_2u_2^\varepsilon(y,t)|\,dydx\\
&\le \left(2\ln\frac{1}{\varepsilon}\right)\theta_i(|a_1|\,\theta_1+|a_2|\,\theta_2).
\end{aligned}$$

On the other hand, since $\varepsilon^2 < 1-\varepsilon^2$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2\times\mathbb{R}^2} \ln^- \frac{1}{|x-y|^2+\varepsilon^2} u_i^\varepsilon(x,t)\,|a_1u_1^\varepsilon(y,t)+a_2u_2^\varepsilon(y,t)|\,dydx\\
&= \int_{|x-y|^2+\varepsilon^2>1} \ln(|x-y|^2+\varepsilon^2) u_i^\varepsilon(x,t)\,|a_1u_1^\varepsilon(y,t)+a_2u_2^\varepsilon(y,t)|\,dydx\\
&\le \int_{|x-y|^2+\varepsilon^2>1} \ln(|x-y|^2+1-\varepsilon^2) u_i^\varepsilon(x,t)\,|a_1u_1^\varepsilon(y,t)+a_2u_2^\varepsilon(y,t)|\,dydx\\
&\le \int_{|x-y|^2+\varepsilon^2>1} \left(\ln 2+\ln|x-y|^2\right) u_i^\varepsilon(x,t)\,(|a_1u_1^\varepsilon(y,t)+a_2u_2^\varepsilon(y,t)|)\}dydx.
\end{aligned}$$

Using the inequality

$$\ln|x-y|^2 \le \ln 2+\ln(1+|x|^2)+\ln(1+|y|^2). \tag{3.64}$$

It is follows that,

$$\begin{aligned}
&\int_{\mathbb{R}^2\times\mathbb{R}^2} \ln^- \frac{1}{|x-y|^2+\varepsilon^2} u_i^\varepsilon(x,t)\,|a_1u_1^\varepsilon(y,t)+a_2u_2^\varepsilon(y,t)|\,dydx\\
&\le \left((2\ln 2)\,\theta_i+\int_{\mathbb{R}^2} u_i^\varepsilon \ln(1+|x|^2)dx\right)(|a_1|\,\theta_1+|a_2|\,\theta_2)\\
&+|a_1|\,\theta_i\int_{\mathbb{R}^2} u_1^\varepsilon\ln(1+|x|^2)dx+|a_2|\,\theta_i\int_{\mathbb{R}^2} u_2^\varepsilon\ln(1+|x|^2)dx.
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_{\mathbb{R}^2} |u_i^\varepsilon v^\varepsilon|\,dx\\
&\le \left(2\ln(2-\varepsilon)\,\theta_i+\int_{\mathbb{R}^2} u_i^\varepsilon \ln(1+|x|^2)dx\right)(|a_1|\,\theta_1+|a_2|\,\theta_2)\\
&+|a_1|\,\theta_i\int_{\mathbb{R}^2} u_1^\varepsilon\ln(1+|x|^2)dx+|a_2|\,\theta_i\int_{\mathbb{R}^2} u_2^\varepsilon\ln(1+|x|^2)dx.
\end{aligned}$$

(viii) Let $2<p<\infty$, Notice on one hand

$$\begin{aligned}
&\||\nabla \mathbf{K}^\varepsilon(x)|\|^p_{L^p(\mathbb{R}^2)}\\
&= \frac{1}{2^{p-1}\pi^{p-1}}\int_0^\infty \frac{r^{p+1}}{(r^2+\varepsilon^2)^p}dr\\
&\le \frac{1}{2^{p-1}\pi^{p-1}}\int_0^1 \frac{r^{p+1}}{(2r\varepsilon)^p}dr+\frac{1}{2^{p-1}\pi^{p-1}}\int_1^\infty \frac{r^{p+1}}{r^{2p}}dr\\
&= \frac{1}{2^{2p}\pi^{p-1}\varepsilon^p}+\frac{1}{(p-2)\,2^{p-1}\pi^{p-1}} =: (C_{10}(\varepsilon,p))^p.
\end{aligned}$$

On the other hand, simple computations show that for $i, j = 1, 2$,

$$\left|\frac{\partial^2 \mathbf{K}^\varepsilon}{\partial x_j \partial x_i}\right| = \left|-\frac{1}{2\pi}\frac{\partial}{\partial x_j}\left(\frac{x_i}{|x|^2+\varepsilon^2}\right)\right| \le \frac{1}{2\pi}\frac{1}{|x|^2+\varepsilon^2}.$$

Therefore

$$\begin{aligned}\left\|\frac{\partial^2 \mathbf{K}^\varepsilon}{\partial x_j \partial x_i}\right\|^p_{L^p(\mathbb{R}^2)} &\le \frac{1}{2^p\pi^{p-1}}\int_0^\infty \frac{2r}{(r^2+\varepsilon^2)^p}dr \\ &= \frac{1}{(p-1)2^p\pi^{p-1}\varepsilon^{2(p-1)}} =: (C_{11}(\varepsilon,p))^p .\end{aligned}$$

Using ([17, Lemma 9.1]), we conclude

$$\frac{\partial \mathbf{K}^\varepsilon}{\partial x_i} * (a_1 u_1^\varepsilon + a_2 u_2^\varepsilon) \in W^{1,p}(\mathbb{R}^2), i = 1, 2,$$

and

$$\begin{aligned}&\frac{\partial}{\partial x_j}\left(\frac{\partial \mathbf{K}^\varepsilon}{\partial x_i} * (a_1 u_1^\varepsilon + a_2 u_2^\varepsilon)\right) \\ &= \frac{\partial^2 \mathbf{K}^\varepsilon}{\partial x_j \partial x_i} * (a_1 u_1^\varepsilon + a_2 u_2^\varepsilon)\, , i, j = 1, 2.\end{aligned}$$

The Young's convolution inequality gives

$$\begin{aligned}&\left\|\frac{\partial v^\varepsilon}{\partial x_i}\right\|_{L^p(\mathbb{R}^2)} \\ &\le \left\|\frac{\partial \mathbf{K}^\varepsilon}{\partial x_i}\right\|_{L^p(\mathbb{R}^2)} \|a_1 u_1^\varepsilon + a_2 u_2^\varepsilon\|_{L^1(\mathbb{R}^2)} \\ &\le \||\nabla \mathbf{K}^\varepsilon|\|_{L^p(\mathbb{R}^2)} \|a_1 u_1^\varepsilon + a_2 u_2^\varepsilon\|_{L^1(\mathbb{R}^2)} \\ &\le C_{10}(\varepsilon,p)(|a_1|\,\theta_1 + |a_2|\,\theta_2) < \infty,\end{aligned}$$

and

$$\begin{aligned}\left\|\frac{\partial^2 v^\varepsilon}{\partial x_j \partial x_i}\right\|_{L^p(\mathbb{R}^2)} &\le \left\|\frac{\partial^2 \mathbf{K}^\varepsilon}{\partial x_j \partial x_i}\right\|_{L^p(\mathbb{R}^2)} \|a_1 u_1^\varepsilon + a_2 u_2^\varepsilon\|_{L^1(\mathbb{R}^2)} \\ &\le C_{11}(\varepsilon,p)(|a_1|\,\theta_1 + |a_2|\,\theta_2) < \infty.\end{aligned}$$

■

Dissipative energy structure We define the free-energy functional $E_\varepsilon(t)$ associated to system (3.49) by

$$\begin{aligned}E_\varepsilon(t) :=& \frac{\mu_1 a_1}{\chi_1 \cos\alpha_1}\int_{\mathbb{R}^2} u_1^\varepsilon \ln u_1^\varepsilon dx + \frac{\mu_2 a_2}{\chi_2 \cos\alpha_2}\int_{\mathbb{R}^2} u_2^\varepsilon \ln u_2^\varepsilon dx \\ &- \frac{a_1}{2}\int_{\mathbb{R}^2} u_1^\varepsilon v^\varepsilon dx - \frac{a_2}{2}\int_{\mathbb{R}^2} u_2^\varepsilon v^\varepsilon dx.\end{aligned} \tag{3.65}$$

In this step, we show that the free energy functional $E_\varepsilon(t)$ enjoys a basic energy law such that it is monotone non-increasing with respect to time.

Theorem 15 *Let $(u_1^\varepsilon, u_2^\varepsilon)$ be a classical solution of system (3.49). Then,*

$$\frac{d}{dt}E_\varepsilon(t) = -\frac{a_1}{\chi_1\cos\alpha_1}\int_{\mathbb{R}^2} u_1^\varepsilon \left|\nabla(\mu_1\ln u_1^\varepsilon - \chi_1\cos\alpha_1 v^\varepsilon)\right|^2 dx - \frac{a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2} u_2^\varepsilon \left|\nabla(\mu_2\ln u_2^\varepsilon - \chi_2\cos\alpha_2 v^\varepsilon)\right|^2 dx, \tag{3.66}$$

for all $t > 0$.
Proof. *Using the first equation of system (3.49) and the decomposition (3.129), we obtain*

$$\partial_t u_1^\varepsilon = \nabla\cdot(u_1^\varepsilon\nabla(\mu_1\ln u_1^\varepsilon - \chi_1\cos\alpha_1 v^\varepsilon)) - \nabla\cdot(\chi_1 u_1^\varepsilon\sin\alpha_1\nabla^\perp v^\varepsilon). \tag{3.67}$$

Multiplying (3.67) by $\mu_1\ln u_1^\varepsilon - \chi_1\cos\alpha_1 v^\varepsilon$ and integrating over $\mathbb{R}^2$

$$\begin{aligned}&\int_{\mathbb{R}^2}(\partial_t u_1^\varepsilon)(\mu_1\ln u_1^\varepsilon - \chi_1\cos\alpha_1 v^\varepsilon)dx\\ &= \int_{\mathbb{R}^2}(\mu_1\ln u_1^\varepsilon - \chi_1\cos\alpha_1 v^\varepsilon)\nabla\cdot(u_1^\varepsilon\nabla(\mu_1\ln u_1^\varepsilon - \chi_1\cos\alpha_1 v^\varepsilon))dx\\ &- \int_{\mathbb{R}^2}(\mu_1\ln u_1^\varepsilon - \chi_1\cos\alpha_1 v^\varepsilon)\nabla\cdot(\chi_1 u_1^\varepsilon\sin\alpha_1\nabla^\perp v^\varepsilon)dx.\end{aligned}$$

Integrating by parts, we have that

$$\begin{aligned}&\int_{\mathbb{R}^2}(\partial_t u_1^\varepsilon)(\mu_1\ln u_1^\varepsilon - \chi_1\cos\alpha_1 v^\varepsilon)dx\\ &= -\int_{\mathbb{R}^2} u_1^\varepsilon\left|\nabla(\mu_1\ln u_1^\varepsilon - \chi_1\cos\alpha_1 v^\varepsilon)\right|^2 dx\\ &+ \int_{\mathbb{R}^2}\left(\mu_1\chi_1\sin\alpha_1\nabla u_1^\varepsilon\cdot\nabla^\perp v^\varepsilon - \chi_1^2 u_1^\varepsilon\sin\alpha_1\cos\alpha_1\nabla v^\varepsilon\cdot\nabla^\perp v^\varepsilon\right)dx\\ &= -\int_{\mathbb{R}^2} u_1^\varepsilon\left|\nabla(\mu_1\ln u_1^\varepsilon - \chi_1\cos\alpha_1 v^\varepsilon)\right|^2 dx\\ &- \mu_1\chi_1\sin\alpha_1\int_{\mathbb{R}^2} u_1^\varepsilon\underbrace{\nabla\cdot\nabla^\perp v^\varepsilon}_{=0}dx - \chi_1^2\sin\alpha_1\cos\alpha_1\int_{\mathbb{R}^2} u_1^\varepsilon\underbrace{\nabla v^\varepsilon\cdot\nabla^\perp v^\varepsilon}_{=0}dx\\ &= -\int_{\mathbb{R}^2} u_1^\varepsilon\left|\nabla(\mu_1\ln u_1^\varepsilon - \chi_1\cos\alpha_1 v^\varepsilon)\right|^2 dx.\end{aligned}$$

In conclusion

$$\int_{\mathbb{R}^2}(\partial_t u_1^\varepsilon)(\mu_1\ln u_1^\varepsilon - \chi_1\cos\alpha_1 v^\varepsilon)dx = -\int_{\mathbb{R}^2} u_1^\varepsilon\left|\nabla(\mu_1\ln u_1^\varepsilon - \chi_1\cos\alpha_1 v^\varepsilon)\right|^2 dx. \tag{3.68}$$

Similarly

$$\int_{\mathbb{R}^2}(\partial_t u_2^\varepsilon)(\mu_2\ln u_2^\varepsilon - \chi_2\cos\alpha_2 v^\varepsilon)dx = -\int_{\mathbb{R}^2} u_2^\varepsilon\left|\nabla(\mu_2\ln u_2^\varepsilon - \chi_2\cos\alpha_2 v^\varepsilon)\right|^2 dx. \tag{3.69}$$

On the other hand, we have that

$$
\begin{aligned}
&\int_{\mathbb{R}^2} (\partial_t u_1^\varepsilon)(\mu_1 \ln u_1^\varepsilon - \chi_1 \cos\alpha_1 v^\varepsilon)dx \\
&= \mu_1 \int_{\mathbb{R}^2} \partial_t (u_1^\varepsilon \ln u_1^\varepsilon - u_1^\varepsilon)dx - \chi_1 \cos\alpha_1 \int_{\mathbb{R}^2} (\partial_t u_1^\varepsilon) v^\varepsilon dx \\
&= \mu_1 \frac{d}{dt} \int_{\mathbb{R}^2} u_1^\varepsilon \ln u_1^\varepsilon dx \qquad (3.70)\\
&- \frac{\chi_1 \cos\alpha_1}{4\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \ln \frac{1}{|x-y|^2+\varepsilon^2} (\partial_t u_1^\varepsilon(x,t))(a_1 u_1^\varepsilon(y,t) + a_2 u_2^\varepsilon(y,t))dydx.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\int_{\mathbb{R}^2} (\partial_t u_2^\varepsilon)(\mu_2 \ln u_2^\varepsilon - \chi_2 \cos\alpha_2 v^\varepsilon)dx = \mu_2 \frac{d}{dt} \int_{\mathbb{R}^2} u_2^\varepsilon \ln u_2^\varepsilon dx \qquad (3.71)\\
&- \frac{\chi_2 \cos\alpha_2}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \frac{1}{|x-y|^2+\varepsilon^2} (\partial_t u_2^\varepsilon(x,t))(a_1 u_1^\varepsilon(y,t) + a_2 u_2^\varepsilon(y,t))dydx.
\end{aligned}
$$

The expression $\frac{a_1}{\chi_1 \cos\alpha_1}$*(3.70)+*$\frac{a_2}{\chi_2 \cos\alpha_2}$*(3.71) gives*

$$
\begin{aligned}
&\frac{a_1}{\chi_1 \cos\alpha_1} \int_{\mathbb{R}^2} (\partial_t u_1^\varepsilon)(\mu_1 \ln u_1^\varepsilon - \chi_1 \cos\alpha_1 v^\varepsilon)dx \\
&+ \frac{a_2}{\chi_2 \cos\alpha_2} \int_{\mathbb{R}^2} (\partial_t u_2^\varepsilon)(\mu_2 \ln u_2^\varepsilon - \chi_2 \cos\alpha_2 v^\varepsilon)dx \\
&= \frac{\mu_1 a_1}{\chi_1 \cos\alpha_1} \frac{d}{dt} \int_{\mathbb{R}^2} u_1^\varepsilon \ln u_1^\varepsilon dx + \frac{\mu_2 a_2}{\chi_2 \cos\alpha_2} \frac{d}{dt} \int_{\mathbb{R}^2} u_2^\varepsilon \ln u_2^\varepsilon dx \\
&- \frac{1}{4\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \ln \frac{1}{|x-y|^2+\varepsilon^2} \partial_t (a_1 u_1^\varepsilon(x,t) + a_2 u_2^\varepsilon(x,t)) \\
&(a_1 u_1^\varepsilon(y,t) + a_2 u_2^\varepsilon(y,t))dydx \\
&= \frac{\mu_1 a_1}{\chi_1 \cos\alpha_1} \frac{d}{dt} \int_{\mathbb{R}^2} u_1^\varepsilon \ln u_1^\varepsilon dx + \frac{\mu_2 a_2}{\chi_2 \cos\alpha_2} \frac{d}{dt} \int_{\mathbb{R}^2} u_2^\varepsilon \ln u_2^\varepsilon dx \\
&- \frac{1}{8\pi} \frac{d}{dt} \int_{\mathbb{R}^2\times\mathbb{R}^2} \ln \frac{1}{|x-y|^2+\varepsilon^2} (a_1 u_1^\varepsilon(x,t) + a_2 u_2^\varepsilon(x,t)) \\
&(a_1 u_1^\varepsilon(y,t) + a_2 u_2^\varepsilon(y,t))dydx.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{d}{dt} \Bigg(\frac{a_1}{\chi_1 \cos\alpha_1} \int_{\mathbb{R}^2} u_1^\varepsilon \ln u_1^\varepsilon dx + \frac{a_2}{\chi_2 \cos\alpha_2} \int_{\mathbb{R}^2} u_2^\varepsilon \ln u_2^\varepsilon dx \\
&- \frac{1}{8\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \ln \frac{1}{|x-y|^2+\varepsilon^2} (a_1 u_1^\varepsilon(x,t) + a_2 u_2^\varepsilon(x,t)) \\
&(a_1 u_1^\varepsilon(y,t) + a_2 u_2^\varepsilon(y,t))dydx \Big) \\
&= -\frac{a_1}{\chi_1 \cos\alpha_1} \int_{\mathbb{R}^2} u_1^\varepsilon \left| \nabla (\mu_1 \ln u_1^\varepsilon - \chi_1 \cos\alpha_1 v^\varepsilon) \right|^2 dx \\
&- \frac{a_2}{\chi_2 \cos\alpha_2} \int_{\mathbb{R}^2} u_2^\varepsilon \left| \nabla (\mu_2 \ln u_2^\varepsilon - \chi_2 \cos\alpha_2 v^\varepsilon) \right|^2 dx.
\end{aligned}
$$

We conclude

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\mu_1 a_1}{\chi_1 \cos\alpha_1}\int_{\mathbb{R}^2} u_1^\varepsilon \ln u_1^\varepsilon dx + \frac{\mu_2 a_2}{\chi_2 \cos\alpha_2}\int_{\mathbb{R}^2} u_2^\varepsilon \ln u_2^\varepsilon dx\right.\\
&\left.-\frac{a_1}{2}\int_{\mathbb{R}^2} u_1^\varepsilon v^\varepsilon dx - \frac{a_2}{2}\int_{\mathbb{R}^2} u_2^\varepsilon v^\varepsilon dx\right)\\
&= -\frac{a_1}{\chi_1 \cos\alpha_1}\int_{\mathbb{R}^2} u_1^\varepsilon \left|\nabla(\mu_1 \ln u_1^\varepsilon - \chi_1 \cos\alpha_1 v^\varepsilon)\right|^2 dx\\
&- \frac{a_2}{\chi_2 \cos\alpha_2}\int_{\mathbb{R}^2} u_2^\varepsilon \left|\nabla(\mu_2 \ln u_2^\varepsilon - \chi_2 \cos\alpha_2 v^\varepsilon)\right|^2 dx,
\end{aligned}
$$

which is equivalent to (3.66). ■

In the remaining of this chapter, we denote by φ_1 a cut-off function in the space $C_0^\infty(\mathbb{R}^2)$ such that $0 \le \varphi_1 \le 1$ and

$$
\varphi_1(x) = \begin{cases} 1 & \text{if} \quad |x| \le 1, \\ 0 & \text{if} \quad |x| \ge 2. \end{cases}
$$

We define furthermore the sequence

$$
\varphi_R(x) := \varphi_1(x/R), \tag{3.72}
$$

which satisfies $\varphi_R(x) \to 1$ as $R \to \infty$. We also notice that for the constant $C^* := \max\{\||\nabla\varphi_1|\|_{L^\infty(\mathbb{R}^2)}, \|\Delta\varphi_1\|_{L^\infty(\mathbb{R}^2)}\}$, we have $|\nabla\varphi_R(x)| \le \frac{C^*}{R}$ and $|\Delta\varphi_R(x)| \le \frac{C^*}{R^2}$.

Boundedness of $\int_{\mathbb{R}^2} u_i^\varepsilon \ln^+ u_i^\varepsilon dx$ Our goal in this step is to show that the positive part of the corresponding entropy functionals, i.e.,

$$
S^+\left[u_i^\varepsilon\right](t) := \int_{\mathbb{R}^2} u_i^\varepsilon \ln^+ u_i^\varepsilon dx, \text{ with } i = 1, 2;
$$

are bounded on the time interval $(0, T)$ uniformly in ε. The standard procedure for doing this is to use the monotonicity of the free energy functional $E_\varepsilon(t)$ combined with the two-dimensional version of the logarithmic Hardy-Littlewood-Sobolev inequality for systems (See [78]) to get a bound for the entropy functionals

$$
\begin{aligned}
S\left[u_i^\varepsilon\right](t) &:= \int_{\mathbb{R}^2} u_i^\varepsilon \ln u_i^\varepsilon dx = \int_{\mathbb{R}^2} u_i^\varepsilon \ln^+ u_i^\varepsilon dx - \int_{\mathbb{R}^2} u_i^\varepsilon \ln^- u_i^\varepsilon dx\\
&=: S^+\left[u_i^\varepsilon\right](t) - S^-\left[u_i^\varepsilon\right](t), \; i = 1, 2.
\end{aligned}
$$

Therefore, if the functionals $S^-\left[u_i^\varepsilon\right]$ are bounded, then also the functionals $S^+\left[u_i^\varepsilon\right]$ are bounded. The boundedness of the functionals $S^-\left[u_i^\varepsilon\right]$ is usually proved for Keller-Segel-type models using estimates like the one given in (3.61) which in turn depends on the control of the moment. One disadvantage of this approach is that the attempt to implement the second moment leads us to deal with integrals that cannot be estimated due to the strong lack of symmetry caused by the tensor flux. Alternatively, we could try to control the logarithmic moment following an approach like the one used in Proposition 14 item (iv).

However, this technique does not work since it requires uniform estimates of $\|u_i^\varepsilon\|_{L^\infty((0,T);L^{4/3}(\mathbb{R}^2))}$, $i = 1, 2$ which we do not have at this stage of our analysis. In consequence, we follow in this subsection a totally different approach to control the functional $S^+[u_i^\varepsilon]$, which is based in a technique introduced in [46]. The idea consists in modifying the entropy functional arising in the free energy functional E_ε (3.65) by a new one that is lower bounded by a constant that depends only on θ_1 and θ_2.

Let $\delta > 0$ be a any small constant, we introduce the modified free energy E_ε^Γ as follows:

$$E_\varepsilon^\Gamma(t) := \frac{\mu_1 a_1}{\chi_1 \cos\alpha_1}\int_{\mathbb{R}^2} u_1^\varepsilon \Gamma(u_1^\varepsilon)dx + \frac{\mu_2 a_2}{\chi_2 \cos\alpha_2}\int_{\mathbb{R}^2} u_2^\varepsilon \Gamma(u_2^\varepsilon)dx \qquad (3.73)$$
$$- \frac{a_1}{2}\int_{\mathbb{R}^2} u_1^\varepsilon v^\varepsilon dx - \frac{a_2}{2}\int_{\mathbb{R}^2} u_2^\varepsilon v^\varepsilon dx.$$

where Γ is defined as

$$\Gamma(u) = \begin{cases} \ln u, & u \geq \eta; \\ \ln\eta + \eta^{-1}(u-\eta) - \frac{\eta^{-2}}{2}(u-\eta)^2, & u < \eta. \end{cases} \qquad (3.74)$$
$$\eta := \min\left\{1, \frac{\delta}{2(\mu_1 a_1 + \mu_2 a_2)(a_1\theta_1 + a_2\theta_2)}\right\}.$$

The Γ function is chosen such that it matches with $\ln u$ when $u \geq \eta$, but $\ln(\eta + (u-\eta))$ is replaced by its degree two Taylor expansion centred at η when $u < \eta$. The advantage of this modification is that the function Γ is bounded from below by $\ln\eta - \frac{3}{2}$.

It is well known that the minimum of two functions in a Sobolev space remains in the same space (By [4, Corollary 5.8.2]). However, it is not obvious that $\min\{f, K\} \in W^{1,p}(\mathbb{R}^2)$ when $f \in W^{1,p}(\mathbb{R}^2)$ and K is constant since $K \notin W^{1,p}(\mathbb{R}^2)$. For the sake of completeness, we provide in the next lemma a statement that considers this special case which will turn out to be fundamental for this research.

Lemma 16 *Let $f \in W^{1,p}(\mathbb{R}^2)$, $1 \leq p < \infty$. Then, $\min\{f, K\}$ belongs to $W^{1,p}(\mathbb{R}^2)$, and*

$$\nabla(\min\{f, K\}) = 1_{\{f<K\}}\nabla f \text{ a.e.},$$

where $K \in \mathbb{R}$.

Proof. Observe that $\min\{f, K\} = f - (f-K)_+$, and we claim that $(f - K)_+ \in W^{1,p}(\mathbb{R}^2)$ and $\nabla((f-K)_+) = 1_{\{f\geq K\}}\nabla f$. Indeed, It is clear that $(f - K)_+ \in L^p(\mathbb{R}^2)$ because $\min\{f, K\} \leq f \in L^p(\mathbb{R}^2)$. On the other hand, set $g^R = \varphi_R(f-K)$, where $\varphi_R \in C_0^\infty(\mathbb{R}^2)$ is a sequence of cut-off defined as in (3.72). Thus, $g^R \in W^{1,p}(\mathbb{R}^2)$ (By [4, Lemma 5.1.2]). By [40, Theorem 4 p. 130], we also have that $g_+^R \in W^{1,p}(\mathbb{R}^2)$ and

$$\nabla\left(g_+^R\right) = 1_{\{g^R\geq 0\}}\nabla g^R \text{ a.e. on } R^2.$$

It follows easily from the dominated convergence theorem that

$$g_+^R = \varphi_R(f-K)_+ \to (f-K)_+ \text{ in } L^p(\mathbb{R}^2).$$

Moreover, note that

$$\nabla\left(g_+^R\right) = 1_{\{\varphi_R(f-K)\geq 0\}}\nabla\left(\varphi_R\left(f-K\right)\right)$$
$$= 1_{\{f\geq K\}}\varphi_R\nabla f + (f-K)_+\nabla\varphi_R.$$

and

$$\|(f-K)_+\nabla\varphi_R\|_{L^p(\mathbb{R}^2)} \leq \frac{C}{R}\|(f-K)_+\|_{L^p(\mathbb{R}^2)} \to 0.$$

Then, we have that

$$\nabla\left(g_+^R\right) \to 1_{\{f\geq K\}}\nabla f.$$

■

The following theorem shows that, despite the possibility of a slow-growing modified free energy, at most linear growth is possible.

Theorem 17 *Let $(u_1^\varepsilon, u_2^\varepsilon)$ be a solution of system (3.49).Then,*

$$\frac{d}{dt}E_\varepsilon^\Gamma(t) \leq \delta, \tag{3.75}$$

for all $t>0$. Furthermore, the following quantity is bounded:

$$\int_{u_i^\varepsilon<1} u_i^\varepsilon\Gamma(u_i^\varepsilon)dx \geq \left(-\ln\eta^{-1} - \frac{3}{2}\right)\theta_i, \tag{3.76}$$

where $i=1,2$.

Proof. Taking the time derivative of $E_\varepsilon^\Gamma(t)$, we have that

$$\frac{d}{dt}E_\varepsilon^\Gamma(t) = \frac{d}{dt}\left(\frac{\mu_1 a_1}{\chi_1\cos\alpha_1}\int_{\mathbb{R}^2} u_1^\varepsilon\Gamma(u_1^\varepsilon)dx + \frac{\mu_2 a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2} u_2^\varepsilon\Gamma(u_2^\varepsilon)dx\right.$$
$$-\frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\ln\frac{1}{|x-y|^2+\varepsilon^2}(a_1u_1^\varepsilon(x,t)+a_2u_2^\varepsilon(x,t))$$
$$\left.(a_1u_1^\varepsilon(y,t)+a_2u_2^\varepsilon(y,t))dydx\right)$$
$$= \frac{a_1}{\chi_1\cos\alpha_1}\int_{\mathbb{R}^2}(\partial_t u_1^\varepsilon)\left(\mu_1\Gamma(u_1^\varepsilon) - \chi_1\cos\alpha_1 v^\varepsilon\right)dx$$
$$+\frac{a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2}(\partial_t u_2^\varepsilon)\left(\mu_2\Gamma(u_2^\varepsilon) - \chi_2\cos\alpha_2 v^\varepsilon\right)dx$$
$$+\frac{\mu_1 a_1}{\chi_1\cos\alpha_1}\int_{\mathbb{R}^2}u_1^\varepsilon\Gamma'(u_1^\varepsilon)\left(\partial_t u_1^\varepsilon\right)dx + \frac{\mu_2 a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2}u_2^\varepsilon\Gamma'(u_2^\varepsilon)\left(\partial_t u_2^\varepsilon\right)dx.$$

Using that

$$\partial_t u_i^\varepsilon = \nabla\cdot(u_i^\varepsilon\nabla(\mu_i\ln u_i^\varepsilon - \chi_i\cos\alpha_i v^\varepsilon)) - \nabla\cdot(\chi_i u_i^\varepsilon\sin\alpha_i\nabla^\perp v^\varepsilon),$$

for $i = 1, 2$, and Integrating by parts, we get

$$
\begin{aligned}
&\frac{d}{dt}E_\varepsilon^\Gamma(t)\\
&= -\frac{a_1}{\chi_1\cos\alpha_1}\int_{\mathbb{R}^2}\nabla(\mu_1\Gamma(u_1^\varepsilon)-\chi_1\cos\alpha_1 v^\varepsilon)\cdot(u_1^\varepsilon\nabla(\mu_1\ln u_1^\varepsilon-\chi_1\cos\alpha_1 v^\varepsilon))dx\\
&+\frac{a_1}{\chi_1\cos\alpha_1}\int_{\mathbb{R}^2}\nabla(\mu_1\Gamma(u_1^\varepsilon)-\chi_1\cos\alpha_1 v^\varepsilon)\cdot(\chi_1 u_1^\varepsilon\sin\alpha_1\nabla^\perp v^\varepsilon)dx\\
&-\frac{\mu_1 a_1}{\chi_1\cos\alpha_1}\int_{\mathbb{R}^2}\nabla\left(u_1^\varepsilon\Gamma'(u_1^\varepsilon)\right)\cdot(u_1^\varepsilon\nabla(\mu_1\ln u_1^\varepsilon-\chi_1\cos\alpha_1 v^\varepsilon))dx\\
&+\frac{\mu_1 a_1}{\chi_1\cos\alpha_1}\int_{\mathbb{R}^2}\nabla\left(u_1^\varepsilon\Gamma'(u_1^\varepsilon)\right)\cdot(\chi_1 u_1^\varepsilon\sin\alpha_1\nabla^\perp v^\varepsilon)dx\\
&-\frac{a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2}\nabla(\mu_2\Gamma(u_2^\varepsilon)-\chi_2\cos\alpha_2 v^\varepsilon)\cdot(u_2^\varepsilon\nabla(\mu_2\ln u_2^\varepsilon-\chi_2\cos\alpha_2 v^\varepsilon))dx\\
&+\frac{a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2}\nabla(\mu_2\Gamma(u_2^\varepsilon)-\chi_2\cos\alpha_2 v^\varepsilon)\cdot(\chi_2 u_2^\varepsilon\sin\alpha_2\nabla^\perp v^\varepsilon)dx\\
&-\frac{\mu_2 a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2}\nabla\left(u_2^\varepsilon\Gamma'(u_2^\varepsilon)\right)\cdot(u_2^\varepsilon\nabla(\mu_2\ln u_2^\varepsilon-\chi_2\cos\alpha_2 v^\varepsilon))\,dx\\
&+\frac{\mu_2 a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2}\nabla\left(u_2^\varepsilon\Gamma'(u_2^\varepsilon)\right)\cdot\left(\chi_2 u_2^\varepsilon\sin\alpha_2\nabla^\perp v^\varepsilon\right)dx\\
&=:\sum_{i=1}^{8}T_i.
\end{aligned}
$$

To estimate the second term T_2 and the fourth term T_4, we define the following functions:

$$
\xi(u)=\int_0^u s\Gamma'(s)ds,\qquad \varsigma(u)=\int_0^u s^2\Gamma''(s)ds.
$$

Then, we have that

$$
\begin{aligned}
T_2&=\frac{\mu_1 a_1\sin\alpha_1}{\cos\alpha_1}\int_{\mathbb{R}^2}\nabla\xi(u_1^\varepsilon)\cdot\nabla^\perp v^\varepsilon dx-a_1\chi_1\sin\alpha_1\int_{\mathbb{R}^2}u_1^\varepsilon\underbrace{\nabla v^\varepsilon\cdot\nabla^\perp v^\varepsilon}_{=0}dx\\
&=-\frac{\mu_1 a_1\sin\alpha_1}{\cos\alpha_1}\int_{\mathbb{R}^2}\xi(u_1^\varepsilon)\underbrace{\nabla\cdot\nabla^\perp v^\varepsilon}_{=0}dx=0,
\end{aligned}
$$

and

$$
\begin{aligned}
T_4&=\frac{\mu_1 a_1\sin\alpha_1}{\cos\alpha_1}\int_{\mathbb{R}^2}u_1^\varepsilon\nabla\left(u_1^\varepsilon\Gamma'(u_1^\varepsilon)\right)\cdot\nabla^\perp v^\varepsilon dx\\
&=\frac{\mu_1 a_1\sin\alpha_1}{\cos\alpha_1}\left(\int_{\mathbb{R}^2}\nabla\xi(u_1^\varepsilon)\cdot\nabla^\perp v^\varepsilon dx+\int_{\mathbb{R}^2}\nabla\varsigma(u_1^\varepsilon)\cdot\nabla^\perp v^\varepsilon dx\right)\\
&=-\frac{\mu_1 a_1\sin\alpha_1}{\cos\alpha_1}\left(\int_{\mathbb{R}^2}\xi(u_1^\varepsilon)\underbrace{\nabla\cdot\nabla^\perp v^\varepsilon}_{=0}dx+\int_{\mathbb{R}^2}\varsigma(u_1^\varepsilon)\underbrace{\nabla\cdot\nabla^\perp v^\varepsilon}_{=0}dx\right)=0.
\end{aligned}
$$

Similarly, we have that $T_6 = T_8 = 0$. On the other hand, simple computations show that

$$
\Gamma'(u)=2\eta^{-1}-\eta^{-2}u\quad\text{for}\quad u\le\eta.
$$

Now we estimate the terms $T_1 + T_3$ as follows:

$$
\begin{aligned}
&T_1 + T_3 \\
&= -\frac{a_1}{\chi_1 \cos\alpha_1} \int_{u_1^\varepsilon \geq \eta} \nabla(\mu_1 \ln u_1^\varepsilon - \chi_1 \cos\alpha_1 v^\varepsilon) \cdot (u_1^\varepsilon \nabla(\mu_1 \ln u_1^\varepsilon - \chi_1 \cos\alpha_1 v^\varepsilon))dx \\
&- \frac{a_1}{\chi_1 \cos\alpha_1} \int_{u_1^\varepsilon < \eta} (\mu_1 \Gamma'(u_1^\varepsilon)\nabla u_1^\varepsilon - \chi_1 \cos\alpha_1 \nabla v^\varepsilon) \cdot (\mu_1 \nabla u_1^\varepsilon - \chi_1 \cos\alpha_1 u_1^\varepsilon \nabla v^\varepsilon)dx \\
&- \frac{\mu_1 a_1}{\chi_1 \cos\alpha_1} \int_{u_1^\varepsilon < \eta} \nabla\left(u_1^\varepsilon \Gamma'(u_1^\varepsilon)\right) \cdot (\mu_1 \nabla u_1^\varepsilon - \chi_1 \cos\alpha_1 u_1^\varepsilon \nabla v^\varepsilon)dx \\
&= -\frac{a_1}{\chi_1 \cos\alpha_1} \int_{u_1^\varepsilon \geq \eta} u_1^\varepsilon \left|\nabla(\mu_1 \ln u_1^\varepsilon - \chi_1 \cos\alpha_1 v^\varepsilon)\right|^2 dx \\
&- \frac{\mu_1^2 a_1}{\chi_1 \cos\alpha_1} \int_{u_1^\varepsilon < \eta} (4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon) \left|\nabla u_1^\varepsilon\right|^2 dx \\
&+ \mu_1 a_1 \int_{u_1^\varepsilon < \eta} ((4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon) u_1^\varepsilon \nabla u_1^\varepsilon \cdot \nabla v^\varepsilon dx \\
&- a_1 \chi_1 \cos\alpha_1 \int_{u_1^\varepsilon < \eta} u_1^\varepsilon \left|\nabla v^\varepsilon\right|^2 dx + \mu_1 a_1 \int_{u_1^\varepsilon < \eta} \nabla u_1^\varepsilon \cdot \nabla v^\varepsilon dx.
\end{aligned}
$$

Using the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
T_1 + T_3 \leq & -\frac{a_1}{\chi_1 \cos\alpha_1} \int_{u_1^\varepsilon \geq \eta} u_1^\varepsilon \left|\nabla(\mu_1 \ln u_1^\varepsilon - \chi_1 \cos\alpha_1 v^\varepsilon)\right|^2 dx \\
& - \frac{\mu_1^2 a_1}{\chi_1 \cos\alpha_1} \int_{u_1^\varepsilon < \eta} (4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon) \left|\nabla u_1^\varepsilon\right|^2 dx \\
& + \mu_1 a_1 \int_{u_1^\varepsilon < \eta} (4\eta^{-1} - 3\eta^{-2} u) u_1^\varepsilon \left|\nabla u_1^\varepsilon\right| \left|\nabla v^\varepsilon\right| dx \\
& - a_1 \chi_1 \cos\alpha_1 \int_{u_1^\varepsilon < \eta} u_1^\varepsilon \left|\nabla v^\varepsilon\right|^2 dx + \mu_1 a_1 \int_{u_1^\varepsilon < \eta} \nabla u_1^\varepsilon \cdot \nabla v^\varepsilon dx.
\end{aligned}
$$

Notice that

$$
\sup_{0 \leq u \leq \eta} \sqrt{(4\eta^{-1} - 3\eta^{-2} u)\, u} \leq \frac{2\sqrt{3}}{3} < 2,
$$

which implies,

$$
\begin{aligned}
T_1 + T_3 \leq & -\frac{a_1}{\chi_1 \cos\alpha_1} \int_{u_1^\varepsilon \geq \eta} u_1^\varepsilon \left|\nabla(\mu_1 \ln u_1^\varepsilon - \chi_1 \cos\alpha_1 v^\varepsilon)\right|^2 dx \\
& - \frac{\mu_1^2 a_1}{\chi_1 \cos\alpha_1} \int_{u_1^\varepsilon < \eta} (4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon) \left|\nabla u_1^\varepsilon\right|^2 dx \\
& + \frac{2\sqrt{3}\mu_1 a_1}{3} \int_{u_1^\varepsilon < \eta} \sqrt{(4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon) u_1^\varepsilon} \left|\nabla u_1^\varepsilon\right| \left|\nabla v^\varepsilon\right| dx \\
& - a_1 \chi_1 \cos\alpha_1 \int_{u_1^\varepsilon < \eta} u_1^\varepsilon \left|\nabla v^\varepsilon\right|^2 dx + \mu_1 a_1 \int_{u_1^\varepsilon < \eta} \nabla u_1^\varepsilon \cdot \nabla v^\varepsilon dx.
\end{aligned}
$$

Completing a square using the 2nd, 3rd, 4th terms in the last line, we obtain that

$$T_1+T_3 \le -\frac{a_1}{\chi_1\cos\alpha_1}\int_{u_1^\varepsilon\ge\eta} u_1^\varepsilon\left|\nabla(\mu_1\ln u_1^\varepsilon-\chi_1\cos\alpha_1 v^\varepsilon)\right|^2dx$$
$$-\frac{2\mu_1^2 a_1}{3\chi_1\cos\alpha_1}\int_{u_1^\varepsilon<\eta}(4\eta^{-1}-3\eta^{-2}u_1^\varepsilon)\left|\nabla u_1^\varepsilon\right|^2dx$$
$$-a_1\chi_1\cos\alpha_1\int_{u_1^\varepsilon<\eta}\left(\frac{\mu_1\sqrt{3}}{3\chi_1\cos\alpha_1}\sqrt{(4\eta^{-1}-3\eta^{-2}u_1^\varepsilon)}\left|\nabla u_1^\varepsilon\right|-\sqrt{u_1^\varepsilon}\left|\nabla v^\varepsilon\right|\right)^2dx$$
$$+\mu_1 a_1\int_{u_1^\varepsilon<\eta}\nabla u_1^\varepsilon\cdot\nabla v^\varepsilon dx.$$

Next,we have that

$$\begin{aligned}\mu_1 a_1\int_{u_1^\varepsilon<\eta}\nabla u_1^\varepsilon\cdot\nabla v^\varepsilon dx &= \mu_1 a_1\int_{\mathbb{R}^2}\nabla(\min\{u_1^\varepsilon,\eta\})\cdot\nabla v^\varepsilon dx\\ &= \mu_1 a_1\int_{\mathbb{R}^2}\min\{u_1^\varepsilon,\eta\}(-\Delta v^\varepsilon)dx\\ &\le \mu_1 a_1\eta\int_{\mathbb{R}^2}(-\triangle\mathbf{K}^\varepsilon*(a_1u_1^\varepsilon+a_2u_2^\varepsilon))\,dx\\ &\le \mu_1 a_1\eta\left\|\triangle\mathbf{K}^\varepsilon\right\|_{L^1}(a_1\theta_1+a_2\theta_2)\le\frac{\delta}{2}.\end{aligned}$$

Here we have applied that $\min\{u_1^\varepsilon,\eta\}\in W^{1,p}(R^2)$ and $\nabla(\min\{u_1^\varepsilon,\eta\}) = 1_{\{u<\eta\}}\nabla u$ a.e. since $u_1^\varepsilon\in W^{1,p}(\mathbb{R}^2)$, for $1\le p<\infty$ (By Lemma 16). Moreover, to justify the integration by parts, we can use a sequence of functions $\psi_n\in C_0^\infty(\mathbb{R}^2)$ such that $\psi_n\to\min\{u_1^\varepsilon,\eta\}$ in $W^{1,4/3}(\mathbb{R}^2)$.We also notice that $\nabla v^\varepsilon\in W^{1,p}(\mathbb{R}^2)^2$ for $p\in(2,\infty)$. Therefore

$$\int_{\mathbb{R}^2}\nabla\psi_n\cdot\nabla v^\varepsilon dx=-\int_{\mathbb{R}^2}\psi_n\Delta v^\varepsilon dx. \tag{3.77}$$

Now, we can pass to the limit in (3.77) when $n\to\infty$, since

$$\begin{aligned}&\int_{\mathbb{R}^2}(\nabla\psi_n-\nabla(\min\{u_1^\varepsilon,\eta\}))\cdot\nabla v^\varepsilon dx\\ &\le\|\nabla\psi_n-\nabla(\min\{u_1^\varepsilon,\eta\})\|_{L^{4/3}(\mathbb{R}^2)}\||\nabla v^\varepsilon|\|_{L^4(\mathbb{R}^2)}\to 0,\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^2}(\psi_n-\min\{u_1^\varepsilon,\eta\})\,\Delta v^\varepsilon dx\\ &\le\|\psi_n-\min\{u_1^\varepsilon,\eta\}\|_{L^{4/3}(\mathbb{R}^2)}\|\Delta v^\varepsilon\|_{L^4(\mathbb{R}^2)}\to 0.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^2}\nabla(\min\{u_1^\varepsilon,\eta\})\cdot\nabla v^\varepsilon dx=-\int_{\mathbb{R}^2}\min\{u_1^\varepsilon,\eta\}\Delta v^\varepsilon dx.$$

In summary

$$
\begin{aligned}
&T_1 + T_3 \le \frac{\delta}{2} - \frac{a_1}{\chi_1 \cos\alpha_1} \int_{u_1^\varepsilon \ge \eta} u_1^\varepsilon \left|\nabla(\mu_1 \ln u_1^\varepsilon - \chi_1 \cos\alpha_1 v^\varepsilon)\right|^2 dx \\
&- \frac{2\mu_1^2 a_1}{3\chi_1 \cos\alpha_1} \int_{u_1^\varepsilon < \eta} (4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon) \left|\nabla u_1^\varepsilon\right|^2 dx \\
&- a_1 \chi_1 \cos\alpha_1 \int_{u_1^\varepsilon < \eta} \left(\frac{\mu_1 \sqrt{3}}{3\chi_1 \cos\alpha_1} \sqrt{(4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon)} \left|\nabla u_1^\varepsilon\right| - \sqrt{u_1^\varepsilon} \left|\nabla v^\varepsilon\right| \right)^2 dx \\
&\le \frac{\delta}{2}.
\end{aligned}
$$

Similarly, we obtain that $T_5 + T_7 \le \frac{\delta}{2}$.Therefore, the estimate (3.75) follows.

Estimate (3.76) follows from the fact that the function Γ is bounded from below by $-\ln\eta^{-1} - \frac{3}{2} \le 0$. Indeed,

$$
\int_{u_i^\varepsilon < 1} u_i^\varepsilon \Gamma(u_i^\varepsilon) dx \ge \left(-\ln\eta^{-1} - \frac{3}{2}\right) \int_{u_i^\varepsilon < 1} u_i^\varepsilon dx \ge \left(-\ln\eta^{-1} - \frac{3}{2}\right) \theta_i.
$$

■

We now recall the logarithmic Hardy-Littlewood-Sobolev inequality for systems. We define the space

$$
\begin{aligned}
\Gamma_M(R^2) := \Big\{ \widetilde{\rho} = (\widetilde{\rho}_i)_{i\in I} : \widetilde{\rho}_i \ge 0, \int_{\mathbb{R}^2} \widetilde{\rho}_i \left|\ln \widetilde{\rho}_i\right| dx < \infty, \\
\int_{\mathbb{R}^2} \widetilde{\rho}_i = M_i, \int_{\mathbb{R}^2} \widetilde{\rho}_i \ln(1+|x|^2) dx < \infty, \forall i \in I \Big\},
\end{aligned}
$$

where $M = (M_i)_{i\in I}$ is given. Next we define the functional $F : \Gamma_M(\mathbb{R}^2) \longrightarrow \mathbb{R}$ by

$$
F[\widetilde{\rho}] = \sum_{i\in I} \int_{\mathbb{R}^2} \widetilde{\rho}_i \ln \widetilde{\rho}_i dx + \frac{1}{4\pi} \sum_{i,j\in I} a_{ij} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \widetilde{\rho}_i(x) \ln|x-y| \, \widetilde{\rho}_j(y) dx dy,
$$

and the polynomial

$$
\Lambda_J(M) = 8\pi \sum_{i\in J} M_i - \sum_{i,j\in J} a_{ij} M_i M_j, \forall J \subseteq I, J \ne \emptyset.
$$

Then we have the following result.

Theorem 18 *(Logarithmic Hardy-Littlewood-Sobolev inequality for systems) Let $A = (a_{ij})$ a symmetric matrix such that $a_{ij} \ge 0$ for all $i, j \in I$ and $M \in \mathbb{R}_+^n$. Then $\Lambda_I(M) = 0$ and*

$$
\begin{aligned}
&\Lambda_J(M) \ge 0, \text{ for all } J \subseteq I, J \ne \emptyset, \\
&\text{If } \Lambda_J(M) = 0 \text{ for some } J, \text{ then } a_{ij} + \Lambda_{J\setminus\{i\}}(M) > 0, \forall i \in J,
\end{aligned}
$$

are necessary and sufficient conditions for the boundedness from below of F on $\Gamma_M(\mathbb{R}^2)$.

Proof. See [78, Theorem 4]. ■

In order to apply this last theorem in the region of the plane $\theta_1\theta_2$ defined by (3.7), we introduce the next technical lemma

Lemma 19 *Let us assume that* (θ_1, θ_2) *satisfies*

$$\theta_1 < \frac{8\pi\mu_1}{a_1\chi_1\cos\alpha_1},\ \theta_2 < \frac{8\pi\mu_2}{a_2\chi_2\cos\alpha_2},$$
$$\text{and } \frac{8\pi\mu_1 a_1}{\chi_1\cos\alpha_1}\theta_1 + \frac{8\pi\mu_2 a_2}{\chi_2\cos\alpha_2}\theta_2 - (a_1\theta_1 + a_2\theta_2)^2 > 0.$$

Then there are constants $b_1 \in (\chi_1, \infty)$ *and* $b_2 \in (\chi_2, \infty)$ *depending on the parameters* $\theta_1, \theta_2, \chi_1, \chi_2, \mu_1, \mu_2, a_1, a_2, \cos\alpha_1$ *and* $\cos\alpha_2$*, such that*

$$\theta_1 \le \frac{8\pi\mu_1}{a_1 b_1\cos\alpha_1},\ \theta_2 \le \frac{8\pi\mu_2}{a_2 b_2\cos\alpha_2}, \tag{3.78}$$

and

$$\frac{8\pi\mu_1 a_1}{b_1\cos\alpha_1}\theta_1 + \frac{8\pi\mu_2 a_2}{b_2\cos\alpha_2}\theta_2 - (a_1\theta_1 + a_2\theta_2)^2 = 0. \tag{3.79}$$

Proof. By hypothesis $\theta_1 \in \left(0, \frac{8\pi\mu_1}{a_1\chi_1\cos\alpha_1}\right)$ and it is clear that $\theta_1 \in \left(0, \theta_1 + \frac{a_2\theta_2}{2a_1}\right)$.

Then, we have that $\theta_1 \in \left(0, \frac{8\pi\mu_1}{a_1\chi_1\cos\alpha_1}\right) \cap \left(0, \theta_1 + \frac{a_2\theta_2}{2a_1}\right)$ which implies the existence of a constant $s_1 > 0$ satisfying

$$\theta_1 < \frac{8\pi\mu_1}{a_1(\chi_1 + s_1)\cos\alpha_1} < \theta_1 + \frac{a_2\theta_2}{2a_1}. \tag{3.80}$$

Similarly, $\theta_2 \in \left(0, \frac{8\pi\mu_2}{a_2\chi_2\cos\alpha_2}\right) \cap \left(0, \theta_2 + \frac{a_1\theta_1}{2a_2}\right)$ implies the existence of a constant $s_2 > 0$ satisfying

$$\theta_2 < \frac{8\pi\mu_2}{a_2(\chi_2 + s_2)\cos\alpha_2} < \theta_2 + \frac{a_1\theta_1}{2a_2}. \tag{3.81}$$

Let us define the function $f : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ by

$$f(x, y) := \frac{8\pi\mu_1 a_1}{x\cos\alpha_1}\theta_1 + \frac{8\pi\mu_2 a_2}{y\cos\alpha_2}\theta_2 - (a_1\theta_1 + a_2\theta_2)^2.$$

Taking $x = \chi_1 + s_1$ and $y = \chi_2 + s_2$, we obtain

$$\begin{aligned}
&f(\chi_1 + s_1, \chi_2 + s_2)\\
&= \frac{8\pi\mu_1 a_1}{(\chi_1 + s_1)\cos\alpha_1}\theta_1 + \frac{8\pi\mu_2 a_2}{(\chi_2 + s_2)\cos\alpha_2}\theta_2 - (a_1\theta_1 + a_2\theta_2)^2\\
&= a_1^2\left(\frac{8\pi\mu_1}{a_1(\chi_1 + s_1)\cos\alpha_1} - \theta_1\right)\theta_1 + a_2^2\left(\frac{8\pi\mu_2}{a_2(\chi_2 + s_2)\cos\alpha_2} - \theta_2\right)\theta_2\\
&- 2a_1a_2\theta_1\theta_2.
\end{aligned}$$

Then, an application of the right side of (3.80) and (3.81), respectively give us

$$\begin{aligned} & f(\chi_1+s_1,\chi_2+s_2) \\ & < \frac{a_1a_2\theta_1\theta_2}{2}+\frac{a_1a_2\theta_1\theta_2}{2}-2a_1a_2\theta_1\theta_2=-a_1a_2\theta_1\theta_2<0. \end{aligned} \tag{3.82}$$

Let us now define the function $g:\mathbb{R}^+\longrightarrow\mathbb{R}$ by

$$g(\tau):=f(\chi_1+s_1\tau,\chi_2+s_2\tau).$$

Note that

$$g(0)=\frac{8\pi\mu_1a_1}{\chi_1\cos\alpha_1}\theta_1+\frac{8\pi\mu_2a_2}{\chi_2\cos\alpha_2}\theta_2-(a_1\theta_1+a_2\theta_2)^2>0,$$

and from (3.82) we get $g(1)<0$. Thus for some $\tau^*\in(0,1)$ it holds $g(\tau^*)=0$. Let us call $b_1:=\chi_1+s_1\tau^*$ and $b_2:=\chi_2+s_2\tau^*$. From the left side of (3.80)

$$\theta_1<\frac{8\pi\mu_1}{a_1(\chi_1+s_1)\cos\alpha_1}<\frac{8\pi\mu_1}{a_1(\chi_1+s_1\tau^*)\cos\alpha_1}=\frac{8\pi\mu_1}{a_1b_1\cos\alpha_1}.$$

Similarly from (3.81)

$$\theta_2<\frac{8\pi\mu_2}{a_2b_2\cos\alpha_2}.$$

The inequality (3.79) follows from $g(\tau^*)=0$. ■

Theorem 20 *Consider a non-negative solution of (3.49) such that* $u_i^\varepsilon\ln(1+|x|^2), u_i^\varepsilon\ln u_i^\varepsilon\in L^\infty_{loc}(\mathbb{R}^+,L^1(\mathbb{R}^2))$ *for* $i=1,2$. *If* (θ_1,θ_2) *satisfies*

$$\begin{aligned} & \theta_1<\frac{8\pi\mu_1}{a_1\chi_1\cos\alpha_1},\ \theta_2<\frac{8\pi\mu_2}{a_2\chi_2\cos\alpha_2}, \\ \text{and } & \frac{8\pi\mu_1a_1}{\chi_1\cos\alpha_1}\theta_1+\frac{8\pi\mu_2a_2}{\chi_2\cos\alpha_2}\theta_2-(a_1\theta_1+a_2\theta_2)^2>0, \end{aligned} \tag{3.83}$$

then for any real $\delta>0$, *there exists a constant* $C_{S^+}:=C(\delta)$ *such that*

$$\int_{\mathbb{R}^2}u_i^\varepsilon(x,t)\ln^+u_i^\varepsilon(x,t)dx\le C_{S^+}+\delta T,\ \text{for any } t\in[0,T], \tag{3.84}$$

where $i=1,2$.

Proof. From (3.75) we have that

$$E_\varepsilon^\Gamma(t)\le E^\Gamma(0)+\delta t,\ \text{for any } t>0.$$

Hence, we estimate the following

$$\begin{aligned} & \frac{\mu_1a_1}{\chi_1\cos\alpha_1}\int_{\mathbb{R}^2}u_1^\varepsilon(x,t)\Gamma(u_1^\varepsilon(x,t))dx+\frac{\mu_2a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2}u_2^\varepsilon(x,t)\Gamma(u_2^\varepsilon(x,t))dx \\ & \le E^\Gamma(0)+\delta t-\frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}a_1u_1^\varepsilon(x,t)a_1u_1^\varepsilon(y,t)\ln|x-y|dxdy \\ & -\frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}a_1u_1^\varepsilon(x,t)a_2u_2^\varepsilon(y,t)\ln|x-y|dxdy \\ & -\frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}a_2u_2^\varepsilon(x,t)a_1u_1^\varepsilon(y,t)\ln|x-y|dxdy \\ & -\frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}a_2u_2^\varepsilon(x,t)a_2u_2^\varepsilon(y,t)\ln|x-y|dxdy. \end{aligned}$$

Applying the definition of Γ (3.74) and (3.76) we get

$$\begin{aligned}
&\frac{\mu_1 a_1}{\chi_1 \cos\alpha_1}\int_{\mathbb{R}^2} u_1^\varepsilon(x,t)\ln^+ u_1^\varepsilon(x,t)dx + \frac{\mu_2 a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2} u_2^\varepsilon(x,t)\ln^+ u_2^\varepsilon(x,t)dx \\
&\le E^\Gamma(0) + \delta t + \frac{\mu_1 a_1\theta_1}{\chi_1\cos\alpha_1}\left(\ln\eta^{-1}+\frac{3}{2}\right) + \frac{\mu_2 a_2\theta_2}{\chi_2\cos\alpha_2}\left(\ln\eta^{-1}+\frac{3}{2}\right) \\
&-\frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} a_1u_1^\varepsilon(x,t)a_1u_1^\varepsilon(y,t)\ln|x-y|dxdy \\
&-\frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} a_1u_1^\varepsilon(x,t)a_2u_2^\varepsilon(y,t)\ln|x-y|dxdy \\
&-\frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} a_2u_2^\varepsilon(x,t)a_1u_1^\varepsilon(y,t)\ln|x-y|dxdy \\
&-\frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} a_2u_2^\varepsilon(x,t)a_2u_2^\varepsilon(y,t)\ln|x-y|dxdy.
\end{aligned}$$

In the next step, positive parameters b_1 and b_2 are introduced in the following way

$$\begin{aligned}
&\frac{\mu_1 a_1}{\chi_1 \cos\alpha_1}\int_{\mathbb{R}^2} u_1^\varepsilon(x,t)\ln^+ u_1^\varepsilon(x,t)dx + \frac{\mu_2 a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2} u_2^\varepsilon(x,t)\ln^+ u_2^\varepsilon(x,t)dx \\
&\le E^\Gamma(0) + \delta t + \frac{\mu_1 a_1\theta_1}{\chi_1\cos\alpha_1}\left(\ln\eta^{-1}+\frac{3}{2}\right) + \frac{\mu_2 a_2\theta_2}{\chi_2\cos\alpha_2}\left(\ln\eta^{-1}+\frac{3}{2}\right) \\
&-\frac{b_1^2\cos^2\alpha_1}{\mu_1^2 4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mu_1 a_1u_1^\varepsilon(x,t)}{b_1\cos\alpha_1}\frac{\mu_1 a_1u_1^\varepsilon(y,t)}{b_1\cos\alpha_1}\ln|x-y|dxdy \\
&-\frac{b_1b_2\cos\alpha_1\cos\alpha_2}{\mu_1\mu_2 4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mu_1 a_1u_1^\varepsilon(x,t)}{b_1\cos\alpha_1}\frac{\mu_2 a_2u_2^\varepsilon(y,t)}{b_2\cos\alpha_2}\ln|x-y|dxdy \\
&-\frac{b_1b_2\cos\alpha_1\cos\alpha_2}{\mu_1\mu_2 4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mu_2 a_2u_2^\varepsilon(x,t)}{b_2\cos\alpha_2}\frac{\mu_1 a_1u_1^\varepsilon(y,t)}{b_1\cos\alpha_1}\ln|x-y|dxdy \\
&-\frac{b_2^2\cos^2\alpha_2}{\mu_2^2 4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mu_2 a_2u_2^\varepsilon(x,t)}{b_2\cos\alpha_2}\frac{\mu_2 a_2u_2^\varepsilon(y,t)}{b_1\cos\alpha_2}\ln|x-y|dxdy. \qquad (3.85)
\end{aligned}$$

Now, we can apply (18) to the functions $\frac{\mu_1 a_1 u_1^\varepsilon}{b_1\cos\alpha_1}$ and $\frac{\mu_2 a_2 u_2^\varepsilon}{b_2\cos\alpha_2}$ in right side of (3.85) getting that

$$\begin{aligned}
&\frac{\mu_1 a_1}{\chi_1 \cos\alpha_1}\int_{\mathbb{R}^2} u_1^\varepsilon(x,t)\ln^+ u_1^\varepsilon(x,t)dx + \frac{\mu_2 a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2} u_2^\varepsilon(x,t)\ln^+ u_2^\varepsilon(x,t)dx \\
&\le E^\Gamma(0) + \delta t + \frac{\mu_1 a_1\theta_1}{\chi_1\cos\alpha_1}\left(\ln\eta^{-1}+\frac{3}{2}\right) + \frac{\mu_2 a_2\theta_2}{\chi_2\cos\alpha_2}\left(\ln\eta^{-1}+\frac{3}{2}\right) - C_{HLS} \\
&+\int_{\mathbb{R}^2}\frac{\mu_1 a_1u_1^\varepsilon(x,t)}{b_1\cos\alpha_1}\ln\left(\frac{\mu_1 a_1u_1^\varepsilon(x,t)}{b_1\cos\alpha_1}\right)dx \\
&+\int_{\mathbb{R}^2}\frac{\mu_2 a_2u_2^\varepsilon(x,t)}{b_2\cos\alpha_2}\ln\left(\frac{\mu_2 a_2u_2^\varepsilon(x,t)}{b_2\cos\alpha_2}\right)dx,
\end{aligned}$$

where the conditions for the existence of the constant C_{HLS} given by Logarithmic HLS inequality for systems are

$$\begin{aligned}
&\theta_1 \le \frac{8\pi\mu_1}{a_1b_1\cos\alpha_1},\ \theta_2 \le \frac{8\pi\mu_2}{a_2b_2\cos\alpha_2}, \\
&\text{and } \frac{8\pi\mu_1 a_1}{b_1\cos\alpha_1}\theta_1 + \frac{8\pi\mu_2 a_2}{b_2\cos\alpha_2}\theta_2 - (a_1\theta_1 + a_2\theta_2)^2 = 0.
\end{aligned} \qquad (3.86)$$

In conclusion we have proved that the conditions (3.86) implies

$$\begin{aligned}
&\frac{\mu_1 a_1}{\cos\alpha_1}\left(\frac{1}{\chi_1}-\frac{1}{b_1}\right)\int_{\mathbb{R}^2} u_1^\varepsilon(x,t)\ln^+ u_1^\varepsilon(x,t)dx \\
&+\frac{\mu_2 a_2}{\cos\alpha_2}\left(\frac{1}{\chi_2}-\frac{1}{b_2}\right)\int_{\mathbb{R}^2} u_2^\varepsilon(x,t)\ln^+ u_2^\varepsilon(x,t)dx \\
&\le E^\Gamma(0)+\delta T - C_{HLS}+\frac{\mu_1 a_1\theta_1}{b_1\cos\alpha_1}\left(\ln\left(\frac{\mu_1 a_1}{b_1\cos\alpha_1}\right)+b_1\ln\eta^{-1}+\frac{3}{2}b_1\right) \\
&+\frac{\mu_2 a_2\theta_2}{b_2\cos\alpha_2}\left(\ln\left(\frac{\mu_2 a_2}{b_2\cos\alpha_2}\right)+b_2\ln\eta^{-1}+\frac{3}{2}b_2\right). \qquad (3.87)
\end{aligned}$$

Note that each of the coefficients of the positive part of the entropy functionals in (3.87) are positive providing $b_1 \in (\chi_1,\infty)$ and $b_2 \in (\chi_2,\infty)$. Then, we have that $\int u_i^\varepsilon \ln^+ u_i^\varepsilon dx$ are bounded below for $i=1,2$. The Lemma 19 gives us that the estimate (3.84) holds for the region (3.83). ■

Boundedness of L^p ***norm for*** $1<p<\infty$ The purpose of this step is to obtain estimates of the L^p-norms for $1<p<\infty$ of the variables u_1^ε and u_2^ε independent of the parameter ε.

Proposition 21 *Assume that* $u_{10}, u_{20} \in L^1(\mathbb{R}^2, \ln(1+|x|^2)dx)$, $u_{10}\ln u_{10}$, $u_{20}\ln u_{20} \in L^1(\mathbb{R}^2, dx)$ *and* (θ_1,θ_2) *satisfies*

$$\theta_1 < \frac{8\pi\mu_1}{a_1\chi_1\cos\alpha_1},\ \theta_2 < \frac{8\pi\mu_2}{a_2\chi_2\cos\alpha_2},$$
$$\text{and } \frac{8\pi\mu_1 a_1}{\chi_1\cos\alpha_1}\theta_1+\frac{8\pi\mu_2 a_2}{\chi_2\cos\alpha_2}\theta_2-(a_1\theta_1+a_2\theta_2)^2>0.$$

If u_{10}, u_{20} *are bounded in* $L^p(\mathbb{R}^2)$ *for some* $p\in(1,\infty)$, *.then any solution* $(u_1^\varepsilon, u_2^\varepsilon)$ *of (3.49) is bounded in* $L^\infty_{loc}(\mathbb{R}^+, L^p(\mathbb{R}^2))$.

Proof. In order to control the L^p norm, $p\in(1,\infty)$ of $u_i^\varepsilon, i=1,2$, we decompose it as follows:

$$u_i^\varepsilon = (u_i^\varepsilon - K)_+ + \min\{u_i^\varepsilon, K\},\ K>1.$$

Note that the function $\min\{u_i^\varepsilon, K\} \in L^p(\mathbb{R}^2)$ is bounded in L^p by $K^{p-1}\theta_i$. Indeed,

$$\int_{\mathbb{R}^2}(\min\{u_i^\varepsilon,K\})^p\,dx \le K^{p-1}\int_{\mathbb{R}^2} u_i^\varepsilon dx = K^{p-1}\theta_i.$$

Then, it is enough to estimate the L^p norm of $(u_i^\varepsilon - K)_+$. For this purpose, we define first

$$M_i(K) := \int_{\mathbb{R}^2}(u_i^\varepsilon-K)_+\,dx.$$

Using the fact that $u_i^\varepsilon \ln^+ u_i^\varepsilon$ is bounded in $L^\infty(\mathbb{R}^+_{Loc}, L^1(\mathbb{R}^2))$, we can estimate $M_i(K)$ by

$$\begin{aligned}
M_i(K) &= \int_{u_i^\varepsilon>K}(u_i^\varepsilon-K)\,dx \le \int_{u_i^\varepsilon>K} u_i^\varepsilon dx \le \frac{1}{\ln K}\int_{u_i^\varepsilon>K} u_i^\varepsilon \ln u_i^\varepsilon dx \\
&\le \frac{1}{\ln K}\int_{\mathbb{R}^2} u_i^\varepsilon \ln^+ u_i^\varepsilon dx.
\end{aligned}$$

and choose it arbitrarily small on any given time interval $(0, T)$.

Multiplying the first equation of system (3.49) by $(u_1^\varepsilon - K)_+^{p-1}$ and integrating over $\mathbb{R}^2$, we get

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^p\,dx \\
&= \mu_1\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^{p-1}\triangle u_1^\varepsilon dx - \chi_1\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^{p-1}\nabla\cdot(u_1^\varepsilon A_1\nabla v^\varepsilon)dx \\
&= \mu_1\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^{p-1}\triangle u_1^\varepsilon dx - \chi_1\cos\alpha_1\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^{p-1}\nabla\cdot(u_1^\varepsilon\nabla v^\varepsilon)dx \\
&- \chi_1\sin\alpha_1\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^{p-1}\nabla\cdot(u_1^\varepsilon\nabla^\perp v^\varepsilon)dx =: T_1 + T_2 + T_3. \qquad (3.88)
\end{aligned}$$

Now we estimate each term in the decomposition (3.88). First, applying the integration by parts and gradient's properties, we have that

$$\begin{aligned}
T_1 &= -\mu_1\int_{\mathbb{R}^2}\nabla\left((u_1^\varepsilon - K)_+^{p-1}\right)\cdot\nabla u_1^\varepsilon dx \\
&= -\mu_1\int_{\mathbb{R}^2}\nabla\left((u_1^\varepsilon - K)_+^{p-1}\right)\cdot\nabla(u_1^\varepsilon - K)dx \\
&= -\frac{4(p-1)\mu_1}{p^2}\int_{\mathbb{R}^2}\left|\nabla\left((u_1^\varepsilon - K)_+^{p/2}\right)\right|^2 dx. \qquad (3.89)
\end{aligned}$$

Second, the identity $\nabla\cdot\nabla^\perp v^\varepsilon = 0$, gradient's properties and integration by parts yield the vanishing of the third term T_3, i.e.,

$$\begin{aligned}
T_3 &= -\chi_1\sin\alpha_1\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^{p-1}\left[\nabla u_1^\varepsilon\cdot\nabla^\perp v^\varepsilon + u_1^\varepsilon\nabla\cdot\nabla^\perp v^\varepsilon\right]dx \\
&= \frac{-\chi_1\sin\alpha_1}{p}\int_{\mathbb{R}^2}\nabla(u_1^\varepsilon - K)_+^p\cdot\nabla^\perp v^\varepsilon dx \\
&= \frac{\chi_1\sin\alpha_1}{p}\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^p\nabla\cdot\nabla^\perp v^\varepsilon dx = 0. \qquad (3.90)
\end{aligned}$$

Now we estimate the second term T_2 as follows:

$$\begin{aligned}
\frac{1}{\chi_1\cos\alpha_1}T_2 &= -\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^{p-1}\left[\nabla u_1^\varepsilon\cdot\nabla v^\varepsilon + u_1^\varepsilon\triangle v^\varepsilon\right]dx \\
&= -\frac{1}{p}\int_{\mathbb{R}^2}\nabla(u_1^\varepsilon - K)_+^p\cdot\nabla v^\varepsilon dx - \int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^{p-1}u_1^\varepsilon\triangle v^\varepsilon dx \\
&= -\frac{1}{p}\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^p(-\triangle v^\varepsilon)dx + \int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^{p-1}u_1^\varepsilon(-\triangle v^\varepsilon)dx.
\end{aligned}$$

Next, we use $-\triangle v^\varepsilon = -\triangle \mathbf{K}^\varepsilon * (a_1 u_1^\varepsilon + a_2 u_2^\varepsilon)$ to obtain

$$
\begin{aligned}
\frac{1}{\chi_1 \cos\alpha_1} T_2 &= -\frac{a_1}{p}\int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p \left(-\triangle \mathbf{K}^\varepsilon * u_1^\varepsilon\right) dx \\
&- \frac{a_2}{p}\int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p \left(-\triangle \mathbf{K}^\varepsilon * u_2^\varepsilon\right) dx \\
&+ a_1 \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} u_1^\varepsilon (-\triangle \mathbf{K}^\varepsilon * u_1^\varepsilon) dx \\
&+ a_2 \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} u_1^\varepsilon (-\triangle \mathbf{K}^\varepsilon * u_2^\varepsilon) dx \\
&= \frac{(p-1)a_1}{p}\int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p \left(-\triangle \mathbf{K}^\varepsilon * u_1^\varepsilon\right) dx \\
&+ \frac{(p-1)a_2}{p}\int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p \left(-\triangle \mathbf{K}^\varepsilon * u_2^\varepsilon\right) dx \\
&+ a_1 K \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} \left(-\triangle \mathbf{K}^\varepsilon * u_1^\varepsilon\right) dx \\
&+ a_2 K \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} \left(-\triangle \mathbf{K}^\varepsilon * u_2^\varepsilon\right) dx.
\end{aligned}
$$

Using the fact that $(-\triangle \mathbf{K}^\varepsilon * K) = K \left\|-\triangle \mathbf{K}^\varepsilon\right\|_{L^1} = K$, we have

$$
\begin{aligned}
\frac{1}{\chi_1 \cos\alpha_1} T_2 &= \tfrac{(p-1)a_1}{p}\int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p \left(-\triangle \mathbf{K}^\varepsilon * (u_1^\varepsilon - K)\right) dx \\
&+ \frac{(p-1)(a_1+a_2)}{p} K \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p dx \\
&+ a_1 K \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} \left(-\triangle \mathbf{K}^\varepsilon * (u_1^\varepsilon - K)\right) dx \\
&+ (a_1 + a_2) K^2 \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} dx \\
&+ \frac{(p-1)a_2}{p}\int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p \left(-\triangle \mathbf{K}^\varepsilon * (u_2^\varepsilon - K)\right) dx \\
&+ a_2 K \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} \left(-\triangle \mathbf{K}^\varepsilon * (u_2^\varepsilon - K)\right) dx.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\frac{1}{\chi_1 \cos\alpha_1} T_2 &\le \frac{(p-1)a_1}{p}\int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p \left(-\triangle \mathbf{K}^\varepsilon * (u_1^\varepsilon - K)_+\right) dx \\
&+ \frac{(p-1)(a_1+a_2)}{p} K \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p dx \\
&+ a_1 K \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} \left(-\triangle \mathbf{K}^\varepsilon * (u_1^\varepsilon - K)_+\right) dx \\
&+ (a_1 + a_2) K^2 \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} dx \\
&+ \frac{(p-1)a_2}{p}\int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p \left(-\triangle \mathbf{K}^\varepsilon * (u_2^\varepsilon - K)_+\right) dx \\
&+ a_2 K \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} \left(-\triangle \mathbf{K}^\varepsilon * (u_2^\varepsilon - K)_+\right) dx.
\end{aligned}
$$

Now we recall the following Young's convolution inequality: Let $r, r' \geq 1$ and $1/r + 1/r' = 1$, if $f \in L^1(\mathbb{R}^2), g \in L^r(\mathbb{R}^2)$ and $h \in L^{r'}(\mathbb{R}^2)$ then

$$\|h\,(f * g)\|_{L^1(\mathbb{R}^2)} \leq \|f\|_{L^1(\mathbb{R}^2)} \|g\|_{L^r(\mathbb{R}^2)} \|h\|_{L^{r'}(\mathbb{R}^2)} . \tag{3.91}$$

Letting $r = p+1$, $f = -\triangle \mathbf{K}^\varepsilon, g = (u_1^\varepsilon - K)_+$ and $h = (u_1^\varepsilon - K)_+^p$, we get

$$\int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p \left(-\triangle \mathbf{K}^\varepsilon * (u_1^\varepsilon - K)_+\right) dx \leq \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p+1}\, dx.$$

Combining the inequality (3.91) but with $g = (u_2^\varepsilon - K)_+$ and the Young's inequality for products we have that

$$\begin{aligned}&\int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p \left(-\triangle \mathbf{K}^\varepsilon * (u_2^\varepsilon - K)_+\right) dx\\ &\leq \frac{p}{p+1} \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p+1} dx + \frac{1}{p+1} \int_{\mathbb{R}^2} (u_2^\varepsilon - K)_+^{p+1}\, dx.\end{aligned}$$

So, we obtain that

$$\begin{aligned}\frac{1}{\chi_1 \cos\alpha_1} T_2 &\leq \frac{(a_1(p+1) + a_2 p)\,(p-1)}{p(p+1)} \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p+1}\, dx\\ &+ \frac{(a_1(2p-1) + 2a_2(p-1))}{p} K \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p dx\\ &+ (a_1 + a_2) K^2 \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} dx\\ &+ \frac{(p-1)a_2}{p(p+1)} \int_{\mathbb{R}^2} (u_2^\varepsilon - K)_+^{p+1}\, dx + \frac{a_2}{p} K \int_{\mathbb{R}^2} (u_2^\varepsilon - K)_+^p\, dx.\end{aligned} \tag{3.92}$$

Substituting (3.89), (3.90) and (3.92) into (3.88), we get

$$\begin{aligned}\frac{1}{p}\frac{d}{dt} \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p\, dx &\leq -\frac{4(p-1)}{p^2} \mu_1 \int_{\mathbb{R}^2} \left|\nabla \left((u_1^\varepsilon - K)_+^{p/2}\right)\right|^2 dx\\ &+ \frac{(a_1(p+1) + a_2 p)\,(p-1)\chi_1 \cos\alpha_1}{p(p+1)} \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p+1} dx\\ &+ \frac{(a_1(2p-1) + 2a_2(p-1))\,\chi_1 \cos\alpha_1}{p} K \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p dx\\ &+ (a_1 + a_2)\chi_1 \cos\alpha_1 K^2 \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} dx\\ &+ \frac{(p-1)a_2 \chi_1 \cos\alpha_1}{p(p+1)} \int_{\mathbb{R}^2} (u_2^\varepsilon - K)_+^{p+1} dx + \frac{a_2 \chi_1 \cos\alpha_1}{p} K \int_{\mathbb{R}^2} (u_2^\varepsilon - K)_+^p dx.\end{aligned} \tag{3.93}$$

Similarly,

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^p\,dx \le -\frac{4(p-1)}{p^2}\mu_2\int_{\mathbb{R}^2}\left|\nabla\left((u_2^\varepsilon-K)_+^{p/2}\right)\right|^2dx \tag{3.94}$$
$$+\frac{(a_2(p+1)+a_1p)\,(p-1)\chi_2\cos\alpha_2}{p(p+1)}\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^{p+1}dx$$
$$+\frac{(a_2(2p-1)+2a_1(p-1))\,\chi_2\cos\alpha_2}{p}K\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^p dx$$
$$+(a_2+a_1)\chi_2\cos\alpha_2K^2\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^{p-1}dx$$
$$+\frac{(p-1)a_1\chi_2\cos\alpha_2}{p(p+1)}\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p+1}dx+\frac{a_1\chi_2\cos\alpha_2}{p}K\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^p dx.$$

The expression $p(3.93)+p(3.94)$ gives

$$\frac{d}{dt}\int_{\mathbb{R}^2}\left((u_1^\varepsilon-K)_+^p+(u_2^\varepsilon-K)_+^p\right)dx \le -\tfrac{4(p-1)}{p}\mu_1\int_{\mathbb{R}^2}\left|\nabla\left((u_1^\varepsilon-K)_+^{p/2}\right)\right|^2dx$$
$$+\frac{(a_1(p+1)+a_2p)\,(p-1)\chi_1\cos\alpha_1+(p-1)a_1\chi_2\cos\alpha_2}{(p+1)}\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p+1}dx$$
$$-\frac{4(p-1)}{p}\mu_2\int_{\mathbb{R}^2}\left|\nabla\left((u_2^\varepsilon-K)_+^{p/2}\right)\right|^2dx$$
$$+\frac{(a_2(p+1)+a_1p)\,(p-1)\chi_2\cos\alpha_2+(p-1)a_2\chi_1\cos\alpha_1}{(p+1)}\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^{p+1}dx$$
$$+\left((a_1(2p-1)+2a_2(p-1))\,\chi_1\cos\alpha_1+a_1\chi_2\cos\alpha_2\right)K\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^p dx$$
$$+\left((a_2(2p-1)+2a_1(p-1))\,\chi_2\cos\alpha_2+a_2\chi_1\cos\alpha_1\right)K\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^p dx$$
$$+(a_1+a_2)p\chi_1\cos\alpha_1K^2\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p-1}dx$$
$$+(a_2+a_1)p\chi_2\cos\alpha_2K^2\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^{p-1}dx.$$

The term involving $\int_{\mathbb{R}^2}(u_i^\varepsilon-K)_+^{p-1}dx$ can be estimated as follows:

$$\begin{aligned}&\int_{\mathbb{R}^2}(u_i^\varepsilon-K)_+^{p-1}dx\\ &\le \int_{K<u_i^\varepsilon\le K+1}1dx+\int_{u_i^\varepsilon>K+1}(u_i^\varepsilon-K)_+^p dx\\ &\le \frac{1}{K}\int_{K<u_i^\varepsilon\le K+1}u_i^\varepsilon dx+\int_{u_i^\varepsilon>K+1}(u_i^\varepsilon-K)_+^p dx\\ &\le \frac{\theta_i}{K}+\int_{\mathbb{R}^2}(u_i^\varepsilon-K)_+^p dx.\end{aligned}$$

Now we recall the following Gagliardo-Nirenberg-Sobolev inequality: Let $1\le p<n$. Then

$$\|w\|_{L^{p^*}}\le C_{GNS}\,\|\nabla w\|_{L^p}\,,\quad \forall w\in W^{1,p}(\mathbb{R}^n). \tag{3.95}$$

Where $C_{GNS} = C(p,n)$ and p^* is given by

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

Letting $p = 1$, $n = 2$, we get that

$$\int_{\mathbb{R}^2} w^2 dx \le C_{GNS}^2 \left(\int_{\mathbb{R}^2} |\nabla w| \, dx \right)^2,$$

or equivalently, with $(u_i^\varepsilon - K)_+^{p+1} = w^2$,

$$\begin{aligned}
\int_{\mathbb{R}^2} (u_i^\varepsilon - K)_+^{p+1} dx &\le C_{GNS}^2 \left(\int_{\mathbb{R}^2} \left| \nabla (u_i^\varepsilon - K)_+^{\frac{p+1}{2}} \right| dx \right)^2 \\
&= K_p \left(\int_{\mathbb{R}^2} \left| (u_i^\varepsilon - K)_+^{1/2} \nabla \left((u_i^\varepsilon - K)_+^{p/2} \right) \right| dx \right)^2 \\
&\le K_p M_i(K) \int_{\mathbb{R}^2} \left| \nabla \left((u_i^\varepsilon - K)_+^{p/2} \right) \right|^2 dx.
\end{aligned}$$

where $K_p := C_{GNS}^2 \left(1 + \frac{1}{p}\right)^2$. So, we have that

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^2} \left((u_1^\varepsilon - K)_+^p + (u_2^\varepsilon - K)_+^p \right) dx \\
&\le -\frac{4(p-1)}{p} \mu_1 \int_{\mathbb{R}^2} \left| \nabla \left((u_1^\varepsilon - K)_+^{p/2} \right) \right|^2 dx \\
&+ \tfrac{(a_1(p+1) + a_2 p)(p-1)\chi_1 \cos\alpha_1 + (p-1) a_1 \chi_2 \cos\alpha_2}{(p+1)} K_p M_1(K) \int_{\mathbb{R}^2} \left| \nabla \left((u_1^\varepsilon - K)_+^{p/2} \right) \right|^2 dx \\
&- \frac{4(p-1)}{p} \mu_2 \int_{\mathbb{R}^2} \left| \nabla \left((u_2^\varepsilon - K)_+^{p/2} \right) \right|^2 dx \\
&+ \tfrac{(a_2(p+1) + a_1 p)(p-1)\chi_2 \cos\alpha_2 + (p-1) a_2 \chi_1 \cos\alpha_1}{(p+1)} K_p M_2(K) \int_{\mathbb{R}^2} \left| \nabla \left((u_2^\varepsilon - K)_+^{p/2} \right) \right|^2 dx \\
&+ (a_1(2p-1) + 2a_2(p-1))\, \chi_1 \cos\alpha_1 + a_1 \chi_2 \cos\alpha_2) K \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p dx \\
&+ (a_1 + a_2) p \chi_1 \cos\alpha_1 K^2 \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p dx \\
&+ \left((a_2(2p-1) + 2a_1(p-1))\, \chi_2 \cos\alpha_2 + a_2 \chi_1 \cos\alpha_1 \right) K \int_{\mathbb{R}^2} (u_2^\varepsilon - K)_+^p dx \\
&+ (a_1 + a_2) p \chi_2 \cos\alpha_2 K^2 \int_{\mathbb{R}^2} (u_2^\varepsilon - K)_+^p dx \\
&+ (a_1 + a_2)(\chi_1 \cos\alpha_1 \theta_1 + \chi_2 \cos\alpha_2 \theta_2) p K.
\end{aligned}$$

By choosing K sufficiently large such that

$$M_i(K) < \frac{4(p-1)(p+1)\mu_i}{p K_p \left((a_i(p+1) + a_j p)\,(p-1)\chi_i \cos\alpha_i + (p-1) a_i \chi_j \cos\alpha_j \right)},$$

for $i, j = 1, 2$, $j \neq i$.Then, for a fixed interval $[0, T]$ with T arbitrarily large

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^2} \left((u_1^\varepsilon - K)_+^p + (u_2^\varepsilon - K)_+^p \right) dx \\
&\le C_{12} \int_{\mathbb{R}^2} \left((u_1^\varepsilon - K)_+^p + (u_2^\varepsilon - K)_+^p \right) dx + C_{13},
\end{aligned}$$

with

$$\begin{array}{rl} C_{12} := & \max((a_1(2p-1)+2a_2(p-1))\,\chi_1\cos\alpha_1 K \\ & +a_1\chi_2\cos\alpha_2 K+(a_1+a_2)p\chi_1\cos\alpha_1 K^1, \\ & +(a_2(2p-1)+2a_1(p-1))\,\chi_2\cos\alpha_2 K \\ & +a_2\chi_1\cos\alpha_1+(a_1+a_2)p\chi_2\cos\alpha_2 K^2 \\ C_{13} := & (a_1+a_2)(\chi_1\cos\alpha_1\theta_1+\chi_2\cos\alpha_2\theta_2)pK. \end{array}$$

By Gronwall's inequality (differential form) [39, p. 624], we have that

$$\begin{aligned} &\int_{\mathbb{R}^2}\left((u_1^\varepsilon-K)_+^p+(u_2^\varepsilon-K)_+^p\right)(x,t)dx \\ &\le e^{C_{12}T}\left(\int_{\mathbb{R}^2}\left((u_{10}-K)_+^p+(u_{20}-K)_+^p\right)dx+C_{13}T\right). \end{aligned}$$

So, we have that $\int_{\mathbb{R}^2}(u_i^\varepsilon-K)_+^p\,dx, i=1,2$ is finite on $[0,T]$ .Therefore, for any $t\in[0,T]$

$$\begin{aligned} &\|u_i^\varepsilon(t)\|_{L^\infty([0,T];L^p(\mathbb{R}^2))} \\ &\le \left\|(u_i^\varepsilon-K)_+\right\|_{L^\infty([0,T];L^p(\mathbb{R}^2))}+\|\min\{u_i^\varepsilon,K\}\|_{L^\infty([0,T];L^p(\mathbb{R}^2))} \\ &\le e^{\frac{C_{12}}{p}T}\left(\int_{\mathbb{R}^2}\left((u_{10}-K)_+^p+(u_{20}-K)_+^p\right)dx+C_{13}T\right)^{\frac{1}{p}}+K^{\frac{p-1}{p}}\theta_i^{\frac{1}{p}}, \end{aligned} \tag{3.96}$$

for any $p\in(1,\infty)$. ■

Extra Uniform estimates

Lemma 22 *Assume that* $0\le u_{10},u_{20}\in L^1(\mathbb{R}^2,\ln(1+|x|^2)dx)\cap L^\infty(\mathbb{R}^2)$, $u_{10}\ln u_{10},\ u_{20}\ln u_{20}\in L^1(\mathbb{R}^2,dx)$ *and* (θ_1,θ_2) *satisfies*

$$\begin{array}{c} \theta_1<\dfrac{8\pi\mu_1}{a_1\chi_1\cos\alpha_1},\ \theta_2<\dfrac{8\pi\mu_2}{a_2\chi_2\cos\alpha_2}, \\ \text{and }\ \dfrac{8\pi\mu_1 a_1}{\chi_1\cos\alpha_1}\theta_1+\dfrac{8\pi\mu_2 a_2}{\chi_2\cos\alpha_2}\theta_2-(a_1\theta_1+a_2\theta_2)^2>0. \end{array}$$

Consider a non-negative solution of (3.49) such that $u_1^\varepsilon,u_2^\varepsilon$ *are bounded in* L^∞_{loc} $(\mathbb{R}^+,L^p(\mathbb{R}^2)),1<p\le\infty$. *Then, with bounds independent on* ε, *we have for all* $T>0$:

(i) *The function* $(t,x)\longmapsto\left|\nabla(u_i^\varepsilon)^{p/2}\right|(x,t)$ *is bounded in* $L^2([0,T];L^2(\mathbb{R}^2))$, *for any* $1\le p<\infty$.

(ii) *The function* $(t,x)\longmapsto|\nabla v^\varepsilon|(x,t)$ *is bounded in* $L^\infty([0,T];L^p(\mathbb{R}^2))$, *for any* $2<p\le\infty$.

(iii) *The function* $(t,x)\longmapsto|u_i^\varepsilon A_1\nabla v^\varepsilon|(x,t),i=1,2,$ *is bounded in* $L^2([0,T];L^2(\mathbb{R}^2))$.

(iv) *The function* $(t,x)\longmapsto u_i^\varepsilon(x,t)\ln(1+|x|^2),i=1,2,$ *is bounded in* $L^\infty([0,T],L^1(\mathbb{R}^2))$.

(v) *The function* $(t,x)\longmapsto u_i^\varepsilon(x,t)\ln u_i^\varepsilon(x,t),i=1,2,$ *is bounded in* $L^\infty([0,T],L^1(\mathbb{R}^2))$.

(vi) *The function* $(t,x) \longmapsto \partial_t u_i^{\varepsilon}(x,t), i = 1,2,$ *is bounded in* $L^2([0,T], H^1(\mathbb{R}^2)^*)$.

(vii) *The function* $(t,x) \longmapsto \sqrt{u_i^{\varepsilon}} |\nabla v^{\varepsilon}| (x,t)$ *is bounded in* $L^2([0,T]; L^2(\mathbb{R}^2))$.

Proof. (i) Assume $p > 1$. Multiplying the first equation of system (3.49) by $(u_1^{\varepsilon})^{p-1}$ and integrating over $\mathbb{R}^2$, we get

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2} |u_1^{\varepsilon}|^p\, dx &= \mu_1 \int_{\mathbb{R}^2} (u_1^{\varepsilon})^{p-1} \triangle u_1^{\varepsilon} dx - \chi_1 \int_{\mathbb{R}^2} (u_1^{\varepsilon})^{p-1} \nabla \cdot (u_1^{\varepsilon} A_1 \nabla v^{\varepsilon}) dx \\
&= \mu_1 \int_{\mathbb{R}^2} (u_1^{\varepsilon})^{p-1} \triangle u_1^{\varepsilon} dx - \chi_1 \cos\alpha_1 \int_{\mathbb{R}^2} (u_1^{\varepsilon})^{p-1} \nabla \cdot (u_1^{\varepsilon} \nabla v^{\varepsilon}) dx \\
&- \chi_1 \sin\alpha_1 \int_{\mathbb{R}^2} (u_1^{\varepsilon})^{p-1} \nabla \cdot (u_1^{\varepsilon} \nabla^{\perp} v^{\varepsilon}) dx \\
&=: T_1 + T_2 + T_3.
\end{aligned} \tag{3.97}$$

Now we estimate T_1 applying the integration by parts and gradient's properties as follows

$$T_1 = -\tfrac{4(p-1)}{p^2}\mu_1 \int_{\mathbb{R}^2} \left|\nabla (u_1^{\varepsilon})^{p/2}\right|^2 dx. \tag{3.98}$$

By the fact that $\nabla \cdot \nabla^{\perp} v^{\varepsilon} = 0$, we have that $T_3 = 0$. Indeed,

$$\begin{aligned}
T_3 &= -\chi_1 \sin\alpha_1 \int_{\mathbb{R}^2} (u_1^{\varepsilon})^{p-1} \left[\nabla u_1^{\varepsilon} \cdot \nabla^{\perp} v^{\varepsilon} + u_1^{\varepsilon} \nabla \cdot \nabla^{\perp} v^{\varepsilon}\right] dx \\
&= \tfrac{-\chi_1 \sin\alpha_1}{p} \int_{\mathbb{R}^2} \nabla (u_1^{\varepsilon})^p \cdot \nabla^{\perp} v^{\varepsilon} dx \\
&= \tfrac{\chi_1 \sin\alpha_1}{p} \int_{\mathbb{R}^2} (u_1^{\varepsilon})^p \nabla \cdot \nabla^{\perp} v^{\varepsilon} dx = 0.
\end{aligned} \tag{3.99}$$

Next we estimate T_2 as follows:

$$\begin{aligned}
T_2 &= -\chi_1 \cos\alpha_1 \int_{\mathbb{R}^2} (u_1^{\varepsilon})^{p-1} \left[\nabla u_1^{\varepsilon} \cdot \nabla v^{\varepsilon} + u_1^{\varepsilon} \triangle v^{\varepsilon}\right] dx \\
&= \tfrac{-\chi_1 \cos\alpha_1}{p} \int_{\mathbb{R}^2} \nabla (u_1^{\varepsilon})^p \cdot \nabla v^{\varepsilon} dx + \chi_1 \cos\alpha_1 \int_{\mathbb{R}^2} (u_1^{\varepsilon})^p (-\triangle v^{\varepsilon})\, dx \\
&= \tfrac{-\chi_1 \cos\alpha_1}{p} \int_{\mathbb{R}^2} (u_1^{\varepsilon})^p (-\triangle v^{\varepsilon}) dx + \chi_1 \cos\alpha_1 \int_{\mathbb{R}^2} (u_1^{\varepsilon})^p (-\triangle v^{\varepsilon})\, dx \\
&= \tfrac{(p-1)\chi_1 \cos\alpha_1}{p} \int_{\mathbb{R}^2} (u_1^{\varepsilon})^p (-\triangle v^{\varepsilon}) dx.
\end{aligned}$$

Next, we use $-\triangle v^{\varepsilon} = -\triangle \mathbf{K}^{\varepsilon} * (a_1 u_1^{\varepsilon} + a_2 u_2^{\varepsilon})$ to obtain

$$\begin{aligned}
T_2 = &\tfrac{(p-1)a_1\chi_1 \cos\alpha_1}{p} \int_{\mathbb{R}^2} (u_1^{\varepsilon})^p (-\triangle \mathbf{K}^{\varepsilon} * u_1^{\varepsilon})\, dx \\
&+ \tfrac{(p-1)a_2\chi_1 \cos\alpha_1}{p} \int_{\mathbb{R}^2} (u_1^{\varepsilon})^p (-\triangle \mathbf{K}^{\varepsilon} * u_2^{\varepsilon})\, dx.
\end{aligned}$$

Using the inequality (3.91) with $r = p+1$, $f = -\triangle \mathbf{K}^{\varepsilon}, g = u_1^{\varepsilon}$ and $h = (u_1^{\varepsilon})^p$, we get

$$\int_{\mathbb{R}^2} (u_1^{\varepsilon})^p (-\triangle \mathbf{K}^{\varepsilon} * u_1^{\varepsilon})\, dx \leq \int_{\mathbb{R}^2} (u_1^{\varepsilon})^{p+1} dx.$$

Combining the inequality (3.91) but with $g = u_2^\varepsilon$ and the Young's inequality for products yields that

$$\int_{\mathbb{R}^2} (u_1^\varepsilon)^p \left(-\triangle \mathbf{K}^\varepsilon * u_2^\varepsilon\right) dx \leq \frac{p}{p+1} \int_{\mathbb{R}^2} (u_1^\varepsilon)^{p+1} dx + \frac{1}{p+1} \int_{\mathbb{R}^2} (u_2^\varepsilon)^{p+1}\, dx.$$

So, we obtain that

$$\begin{aligned} T_2 \leq & \tfrac{(p-1)\chi_1 \cos\alpha_1 (a_1(p+1)+a_2 p)}{p(p+1)} \int_{\mathbb{R}^2} (u_1^\varepsilon)^{p+1} dx \\ & + \tfrac{(p-1)a_2\chi_1\cos\alpha_1}{p(p+1)} \int_{\mathbb{R}^2} (u_2^\varepsilon)^{p+1}\, dx. \end{aligned} \tag{3.100}$$

Substituting (3.98), (3.99) and (3.100) into p(3.97), we get

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^2} |u_1^\varepsilon|^p\, dx \leq & -\tfrac{4(p-1)}{p}\mu_1 \int_{\mathbb{R}^2} \left|\nabla (u_1^\varepsilon)^{p/2}\right|^2 dx \\ & + \tfrac{(p-1)\chi_1\cos\alpha_1(a_1(p+1)+a_2p)}{(p+1)} \int_{\mathbb{R}^2} (u_1^\varepsilon)^{p+1} dx \\ & + \tfrac{(p-1)a_2\chi_1\cos\alpha_1}{(p+1)} \int_{\mathbb{R}^2} (u_2^\varepsilon)^{p+1}\, dx. \end{aligned}$$

From (3.96), we have that

$$\frac{d}{dt}\int_{\mathbb{R}^2} |u_1^\varepsilon|^p\, dx + \tfrac{4(p-1)}{p}\mu_1 \int_{\mathbb{R}^2} \left|\nabla (u_1^\varepsilon)^{p/2}\right|^2 dx \leq C_{14}, \tag{3.101}$$

Integrating (3.101) from 0 to T we have that for any $T > 0$

$$\int_0^T \int_{\mathbb{R}^2} \left|\nabla (u_1^\varepsilon)^{p/2}\right|^2 dx \leq \tfrac{p}{4(p-1)\mu_1} \left(\int_{\mathbb{R}^2} |u_{10}|^p\, dx + C_{14} T\right).$$

Similarly,

$$\int_0^T \int_{\mathbb{R}^2} \left|\nabla (u_2^\varepsilon)^{p/2}\right|^2 dx \leq \tfrac{p}{4(p-1)\mu_2} \left(\int_{\mathbb{R}^2} |u_{20}|^p\, dx + C_{15} T\right).$$

The case $p = 1$ follows from (3.62), (3.63), and item (**v**). Therefore, we get

$$\left\|\left|\nabla (u_i^\varepsilon)^{p/2}\right|\right\|_{L^2([0,T];L^2(\mathbb{R}^2))} \leq C_{16}, \text{ for all } 1 \leq p < \infty, i = 1,2. \tag{3.102}$$

(ii) For $2 < p < \infty$, we have by (3.20) that

$$\| |\nabla v^\varepsilon| \|_{L^p(\mathbb{R}^2)} \leq \frac{C_{17}}{2\pi} \left(a_1 \|u_1^\varepsilon\|_{L^{\frac{2p}{2+p}}(\mathbb{R}^2)} + a_2 \|u_2^\varepsilon\|_{L^{\frac{2p}{2+p}}(\mathbb{R}^2)} \right),$$

and for $p = \infty$, we can use the inequality (3.21) with $q = 3$ to obtain

$$\| |\nabla v^\varepsilon| \|_{L^\infty(\mathbb{R}^2)} \leq \frac{C_{18}}{2\pi} (a_1\theta_1 + a_2\theta_2)^{1/4} \left(a_1 \|u_1^\varepsilon\|_{L^3(\mathbb{R}^2)} + a_2 \|u_2^\varepsilon\|_{L^3(\mathbb{R}^2)} \right)^{3/4}.$$

From (3.96), we have that

$$\| |\nabla v^\varepsilon| \|_{L^p(\mathbb{R}^2)} \leq C_{19}, \text{ for any } 2 < p \leq \infty.$$

(iii) The bound of $|u_i^\varepsilon A_1 \nabla v^\varepsilon| \in L^2([0,T]\,;L^2(\mathbb{R}^2))$ follows by using that u_i^ε is bounded in $L^\infty([0,T]\,;L^2(\mathbb{R}^2))$. Indeed, Using the Holder inequality, we have that

$$\begin{aligned}\int_0^T \||u_1^\varepsilon A_1\nabla v^\varepsilon|\|^2_{L^p(\mathbb{R}^2)}\,dt &\le \int_0^T \||A_1\nabla v^\varepsilon|\|^2_{L^\infty(\mathbb{R}^2)} \|u_i^\varepsilon\|^2_{L^2(\mathbb{R}^2)}\,dt \\ &\le \||\nabla v^\varepsilon|\|^2_{L^\infty([0,T];L^\infty(\mathbb{R}^2))} \|u_i^\varepsilon\|^2_{L^\infty([0,T];L^2(\mathbb{R}^2))} T.\end{aligned}$$

(iv) Recall that

$$\left|\nabla \ln(1+|x|^2)\right| = \frac{2|x|}{1+|x|^2} \le 1, \text{ and } \left|\Delta \ln(1+|x|^2)\right| = \frac{4}{\left(1+|x|^2\right)^2} \le 4.$$

Multiplying the first equation of system (3.49) by $\ln(1+|x|^2)$ and integrating over $\mathbb{R}^2$, we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2} u_1^\varepsilon \ln(1+|x|^2)dx \\ &= \mu_1 \int_{\mathbb{R}^2} u_1^\varepsilon \Delta \ln(1+|x|^2)dx + \chi_1 \int_{\mathbb{R}^2} \nabla \ln(1+|x|^2)\cdot(u_1^\varepsilon A_1 \nabla v^\varepsilon)dx \\ &\le 4\mu_1\theta_1 + \chi_1 \int_{\mathbb{R}^2} u_1^\varepsilon |\nabla v^\varepsilon|\,dx \le 4\mu_1\theta_1 + \chi_1 \||\nabla v^\varepsilon|\|_{L^\infty(\mathbb{R}^2)}\theta_1.\end{aligned}$$

By **(ii)**, we have that

$$\frac{d}{dt}\int_{\mathbb{R}^2} u_1^\varepsilon \ln(1+|x|^2)dx \le C_{20}.$$

Integrating on the interval $(0,t)$, we obtain that

$$\int_{\mathbb{R}^2} u_1^\varepsilon \ln(1+|x|^2)dx \le \int_{\mathbb{R}^2} u_{10} \ln(1+|x|^2)dx + C_{29}T, \text{ for any } t \in [0,T].$$

Similarly,

$$\int_{\mathbb{R}^2} u_2^\varepsilon \ln(1+|x|^2)dx \le \int_{\mathbb{R}^2} u_{20} \ln(1+|x|^2)dx + C_{21}T, \text{ for any } t \in [0,T].$$

(v) The bound of $u_i^\varepsilon |\ln u_i^\varepsilon| \in L^\infty([0,T]\,;L^1(\mathbb{R}^2)), i=1,2,$ follows by using that $u_i^\varepsilon \ln^+ u_i^\varepsilon dx$ and $u_i^\varepsilon \ln(1+|x|^2)$ are bounded in $L^\infty([0,T]\,;L^1(\mathbb{R}^2))$.Indeed, by (3.61) we get

$$\begin{aligned}&\int_{\mathbb{R}^2} u_i^\varepsilon |\ln u_i^\varepsilon|\,dx \\ &\le \int_{\mathbb{R}^2} u_i^\varepsilon \ln^+ u_i^\varepsilon dx + \pi + 4\int_{\mathbb{R}^2} u_i^\varepsilon \ln(1+|x|^2)dx \le C_{22}, i=1,2.\end{aligned}$$

(vi) For any $\psi \in H^1(\mathbb{R}^2)$ with $\|\psi\|_{H^1(\mathbb{R}^2)} \le 1$, we have

$$\begin{aligned}&\langle \partial_t u_i^\varepsilon, \psi\rangle_{(H^1(\mathbb{R}^2)^*,H^1(\mathbb{R}^2))} \\ &= \mu_1 \langle \triangle u_1^\varepsilon, \psi\rangle_{(H^1(\mathbb{R}^2)^*,H^1(\mathbb{R}^2))} \\ &- \chi_1 \langle \nabla\cdot(u_1^\varepsilon A_1 \nabla v^\varepsilon), \psi\rangle_{(H^1(\mathbb{R}^2)^*,H^1(\mathbb{R}^2))}.\end{aligned} \tag{3.103}$$

We observe

$$
\begin{aligned}
&\langle \triangle u_1^\varepsilon, \psi\rangle_{(H^1(\mathbb{R}^2)^*, H^1(\mathbb{R}^2))}\\
&= -\left\langle \nabla u_1^\varepsilon, \nabla\psi\right\rangle_{(H^1(\mathbb{R}^2)^*, H^1(\mathbb{R}^2))}\\
&= -\int_{\mathbb{R}^2} \nabla u_1^\varepsilon \cdot \nabla\psi dx\\
&\leq \||\nabla u_1^\varepsilon|\|_{L^2(\mathbb{R}^2)} \||\nabla\psi|\|_{L^2(\mathbb{R}^2)} \leq \||\nabla u_1^\varepsilon|\|_{L^2(\mathbb{R}^2)}.
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle \nabla\cdot(u_1^\varepsilon A_1 \nabla v^\varepsilon), \psi\rangle_{(H^1(\mathbb{R}^2)^*, H^1(\mathbb{R}^2))}\\
&= -\left\langle u_1^\varepsilon A_1\nabla v^\varepsilon, \nabla\psi\right\rangle_{(H^1(\mathbb{R}^2)^*, H^1(\mathbb{R}^2))} = -\int_{\mathbb{R}^2} u_1^\varepsilon A_1\nabla v^\varepsilon\cdot\nabla\psi dx\\
&\leq \||u_1^\varepsilon A_1\nabla v^\varepsilon|\|_{L^2(\mathbb{R}^2)} \||\nabla\psi|\|_{L^2(\mathbb{R}^2)} \leq \||u_1^\varepsilon A_1\nabla v^\varepsilon|\|_{L^2(\mathbb{R}^2)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_0^T \|\triangle u_1^\varepsilon\|^2_{H^1(\mathbb{R}^2)^*} dt &= \int_0^T \left(\sup_{\|\psi\|_{H^1(\mathbb{R}^2)}\leq 1} \langle \triangle u_1^\varepsilon, \psi\rangle_{(H^1(\mathbb{R}^2)^*, H^1(\mathbb{R}^2))}\right)^2 dt\\
&\leq \int_0^T \||\nabla u_1^\varepsilon|\|^2_{L^2(\mathbb{R}^2)} dt. \qquad (3.104)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T \|\nabla\cdot(u_1^\varepsilon A_1\nabla v^\varepsilon)\|^2_{H^1(\mathbb{R}^2)^*} dt\\
&= \int_0^T \left(\sup_{\|\psi\|_{H^1(\mathbb{R}^2)}\leq 1} \langle \nabla\cdot(u_1^\varepsilon A_1\nabla v^\varepsilon), \psi\rangle_{(H^1(\mathbb{R}^2)^*, H^1(\mathbb{R}^2))}\right)^2 dt\\
&\leq \int_0^T \||u_1^\varepsilon A_1\nabla v^\varepsilon|\|^2_{L^2(\mathbb{R}^2)} dt. \qquad (3.105)
\end{aligned}
$$

Hence (3.103), (3.104) and (3.105) yield us to

$$
\begin{aligned}
&\left(\int_0^T \|\partial_t u_i^\varepsilon\|^2_{H^1(\mathbb{R}^2)^*} dt\right)^{\frac{1}{2}}\\
&\leq \mu_1 \left(\int_0^T \||\triangle u_1^\varepsilon|\|^2_{H^1(\mathbb{R}^2)^*} dt\right)^{\frac{1}{2}} + \chi_1 \left(\int_0^T \||\nabla\cdot(u_1^\varepsilon A_1\nabla v^\varepsilon)|\|^2_{H^1(\mathbb{R}^2)^*} dt\right)^{\frac{1}{2}}\\
&\leq \mu_1 \left(\int_0^T \||\nabla u_1^\varepsilon|\|^2_{L^2(\mathbb{R}^2)} dt\right)^{\frac{1}{2}} + \chi_1 \left(\int_0^T \||u_1^\varepsilon A_1\nabla v^\varepsilon|\|^2_{L^2(\mathbb{R}^2)} dt\right)^{\frac{1}{2}}\\
&\leq \mu_1 \||\nabla u_1^\varepsilon|\|_{L^2([0,T];L^2(\mathbb{R}^2))} + \chi_1 \||u_1^\varepsilon A_1\nabla v^\varepsilon|\|_{L^2([0,T];L^2(\mathbb{R}^2))}.
\end{aligned}
$$

By items **(i)** and **(iii)**, we get

$$
\|\partial_t u_i^\varepsilon\|_{L^2([0,T];H^1(\mathbb{R}^2)^*)} \leq C_{23}.
$$

(**vii**) The bound of $\sqrt{u_i^\varepsilon}\,|\nabla v^\varepsilon| \in L^2([0,T]\,;L^2(\mathbb{R}^2))$ follows follow from item (**ii**) and the conservation of mass property. Indeed, we have that

$$\int_0^T \left\|\sqrt{u_i^\varepsilon}\,|\nabla v^\varepsilon|\right\|_{L^2(\mathbb{R}^2)}^2 dt \leq \int_0^T \||\nabla v^\varepsilon|\|_{L^\infty(\mathbb{R}^2)}^2 \left\|\sqrt{u_i^\varepsilon}\right\|_{L^2(\mathbb{R}^2)}^2 dt$$
$$\leq \||\nabla v^\varepsilon|\|_{L^\infty([0,T];L^\infty(\mathbb{R}^2))}^2\,\theta_i T. \tag{3.106}$$

■

Strong convergence of u_i^ε In order to establish the strong convergence of u_i^ε in $L^2([0,T]\,;L^2(\mathbb{R}^2))$, we will use the Aubin-Lions compactness method (See e.g. [2, Theorem 5.1], [60, Theorem 12.1] and [79, Corollary 4]).

Lemma 23 (Aubin-Lions) *Let X, Y and Z be three Banach spaces with $X \subset Y \subset Z$. Suppose that X is compactly embedded in Y and that Y is continuously embedded in Z. For $1 \leq p \leq q \leq \infty$, let*

$$V = \{u \in L^p([0,T]\,;X) : \partial_t u \in L^q([0,T]\,;Z)\}\,.$$

(i) *If $p < \infty$ then the embedding of V into $L^p([0,T]\,;Y)$ is compact.*

(ii) *If $p = \infty$ and $q > 1$ then the embedding of V into $C([0,T]\,;Y)$ is compact.*

Taking now into account the embedding

$$H^1(\Omega) \overset{\text{Compact}}{\hookrightarrow}{}^5 L^2(\Omega) \overset{\text{Continuous}}{\hookrightarrow} H^1(\Omega)^*,$$

where Ω is a bounded open set of class C^1. By Lemma 23, we have that for any Ω there exists a subsequence,still denoted by $u_i^\varepsilon, i = 1,2$, such that

$$u_i^\varepsilon \to u_i \text{ in } L^2([0,T]\,;L^2(\Omega)).$$

By a diagonal argument, the following uniform strong convergence holds true that for any $R > 0$

$$u_i^\varepsilon \to u_i \text{ in } L^2([0,T]\,;L^2(B_R(0))). \tag{3.107}$$

Now, to extend (3.107) to the whole space, we observe that

$$\int_0^T \|u_i^\varepsilon\|_{L^2(|x|>R)}^2\,dt$$
$$= \int_0^T \int_{|x|>R} (u_i^\varepsilon)^2\,dxdt \leq \frac{1}{\sqrt{\ln(1+R^2)}} \int_0^T \int_{\mathbb{R}^2} \left(\ln(1+|x|^2)\right)^{1/2} (u_i^\varepsilon)^2\,dxdt$$
$$\leq \frac{1}{\sqrt{\ln(1+R^2)}} \int_0^T \left(\|u_i^\varepsilon\|_{L^3(\mathbb{R}^2)}^{3/2} \left(\int_{\mathbb{R}^2} \ln(1+|x|^2) u_i^\varepsilon dx\right)^{1/2}\right) dt \to 0,$$

[5] By Rellich–Kondrachov Theorem [17, Theorem 9.16].

as $R \to \infty$ and the weak semi-continuity of $L^2([0,T]\,;L^2(\mathbb{R}^2))$ implies

$$\begin{aligned}&\int_0^T \|u_i\|^2_{L^2(|x|>R)}\,dt\\ &\le \liminf_{\varepsilon\to 0}\int_0^T \|u_i^\varepsilon\|^2_{L^2(|x|>R)}\,dt \to 0 \text{ as } R\to\infty.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^T \|u_i^\varepsilon - u_i\|^2_{L^2(\mathbb{R}^2)}\,dt\\ &\le 2\int_0^T \left(\|u_i^\varepsilon\|^2_{L^2(|x|>R)} + \|u_i\|^2_{L^2(|x|>R)} + \|u_i^\varepsilon - u_i\|^2_{L^2(|x|\le R)}\right)dt \to 0,\end{aligned}$$

as $R \to \infty, \varepsilon \to 0$. So, we have that

$$u_i^\varepsilon \to u_i \text{ in } L^2([0,T]\,;L^2(\mathbb{R}^2)). \tag{3.108}$$

Proposition 24 *Let (f_n) be a sequence of functions in $L^p([0,T]\,;L^p(\mathbb{R}^2)), 1 \le p \le \infty$ and let $f \in L^p([0,T]\,;L^p(\mathbb{R}^2))$ be such that $\|f_n - f\|_{L^p([0,T];L^p(\mathbb{R}^2))} \to 0$. Then, there is a subsequence (f_{nk}) such that $\|f_{nk} - f\|_{L^p(\mathbb{R}^2)} \to 0$ for a.e. on $[0,T]$.*

Proof. The conclusion is obvious when $p = \infty$, indeed,

$$\|f_n - f\|_{L^\infty(\mathbb{R}^2)} \le \|f_n - f\|_{L^\infty([0,T];L^\infty(\mathbb{R}^2))} \to 0.$$

Thus we assume $1 \le p < \infty$. By [17, Theorem 4.9], we know that there exists a subsequence (f_{nk}) and a function $h \in L^p([0,T]\,;L^p(\mathbb{R}^2))$ such that

$$\begin{aligned}f_{nk}(x,t) &\to f(x,t) \text{ a.e. on } [0,T]\times\mathbb{R}^2,\\ |f_{nk}(x,t)| &\le h(x,t) \text{ for all } k, \text{a.e. on } [0,T]\times\mathbb{R}^2.\end{aligned}$$

Note that

$$\|h\|^p_{L^p([0,T];L^p(\mathbb{R}^2))} = \int_0^T \|h\|^p_{L^p(\mathbb{R}^2)}\,dt = \int_0^T\int_{\mathbb{R}^2} h^p\,dx\,dt < \infty.$$

By Fubini's theorem, we have that $h(x,t) \in L^p(\mathbb{R}^2)$ for a.e. on $[0,T]$. Then, we can apply the Lebesgue's dominated convergence theorem to conclude that

$$\|f_{nk} - f\|_{L^p(\mathbb{R}^2)} \to 0 \text{ for a.e. on } [0,T].$$

■

By Proposition 24, there is subsequence, still denoted by u_i^ε, such that

$$u_i^\varepsilon(t) \to u_i(t) \text{ in } L^2(\mathbb{R}^2) \text{ for a.e. on } [0,T]. \tag{3.109}$$

Mass conservation Multiplying the first equation of system (3.49) by any test function $\varphi(x) \in C_0^\infty(\mathbb{R}^2)$ and integrating over $[0,t) \times \mathbb{R}^2$

$$\begin{aligned}&\int_{\mathbb{R}^2} \varphi u_1^\varepsilon(x,t)dx - \int_{\mathbb{R}^2} \varphi u_{10}(x)dx \\ &= \mu_1 \int_0^t \int_{\mathbb{R}^2} u_1^\varepsilon \Delta\varphi dx d\tau + \chi_1 \int_0^t \int_{\mathbb{R}^2} \nabla\varphi \cdot (u_1^\varepsilon A_1 \nabla v^\varepsilon) dx d\tau.\end{aligned}$$

Letting $\varphi(x) = \varphi_R(x)$ be defined as in (3.72), we have

$$\left| \int_0^t \int_{\mathbb{R}^2} u_1^\varepsilon \Delta\varphi_R dx d\tau \right| \le \frac{C^*}{R^2}\theta_1 T,$$

and

$$\left| \int_0^t \int_{\mathbb{R}^2} \nabla\varphi \cdot (u_1^\varepsilon A_1 \nabla v^\varepsilon) dx d\tau \right| \le \frac{C^*}{R} \|\nabla v^\varepsilon\|_{L^\infty([0,T];L^\infty(\mathbb{R}^2))} \theta_1 T \le \frac{C_{24}}{R}\theta_1 T.$$

Due to (3.109),passing to the limit $\varepsilon \to 0, R \to \infty$, we obtain the mass conservation property

$$\int_{\mathbb{R}^2} u_1(x,t)dx = \theta_1.$$

Similarly,

$$\int_{\mathbb{R}^2} u_2(x,t)dx = \theta_2.$$

Existence of the weak solution Now multiplying the first equation of system (3.49) by any test function $\varphi \in C_0^\infty(\mathbb{R}^2)$ and integrating over $[0,t) \times R^2$, we get the weak formulation for u_1^ε

$$\begin{aligned}&\int_{\mathbb{R}^2} \varphi u_1^\varepsilon(x,t)dx - \int_{\mathbb{R}^2} \varphi u_{10}(x)dx \\ &= \mu_1 \int_0^t \int_{\mathbb{R}^2} u_1^\varepsilon \Delta\varphi dx d\tau + \chi_1 \int_0^t \int_{\mathbb{R}^2} \nabla\varphi \cdot (u_1^\varepsilon A_1 \nabla v^\varepsilon)\, dx d\tau.\end{aligned} \tag{3.110}$$

Notice that (3.108) directly yields

$$\int_0^t \int_{\mathbb{R}^2} u_1^\varepsilon \Delta\varphi dx d\tau \to \int_0^t \int_{\mathbb{R}^2} u_1 \Delta\varphi dx d\tau,\ \varepsilon \to 0. \tag{3.111}$$

In order to deal with the nonlinear term arising in (3.110), we first notice that $|\nabla v^\varepsilon|$ is bounded in $L^4([0,T]\,; L^4(\mathbb{R}^2))$, thus there is subsequence, still denoted by ∇v^ε, and a function $w \in L^4([0,T]\,; L^4(\mathbb{R}^2))^2$ such that

$$\nabla v^\varepsilon \rightharpoonup w \text{ in } L^4([0,T]\,; L^4(\mathbb{R}^2))^2. \tag{3.112}$$

Moreover, applying the interpolation inequality and next the Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^T \|u_i^\varepsilon - u_i\|_{L^{4/3}(\mathbb{R}^2)}^{4/3} dt \\
&\leq \int_0^T \|u_i^\varepsilon - u_i\|_{L^2(\mathbb{R}^2)}^{2/3} \|u_i^\varepsilon - u_i\|_{L^1(\mathbb{R}^2)}^{2/3} dt \\
&\leq \left(\int_0^T \|u_i^\varepsilon - u_i\|_{L^2(\mathbb{R}^2)}^2 dt\right)^{1/3} \left(\int_0^T \|u_i^\varepsilon - u_i\|_{L^1(\mathbb{R}^2)} dt\right)^{2/3} \\
&\leq (2\theta_i T)^{2/3} \left(\int_0^T \|u_i^\varepsilon - u_i\|_{L^2(\mathbb{R}^2)}^2 dt\right)^{1/3}.
\end{aligned}$$

Next, an application of the strong convergence result (3.108) gives

$$\|u_i^\varepsilon - u_i\|_{L^{4/3}([0,T];L^{4/3}(\mathbb{R}^2))} \leq \sqrt{2\theta_i T}\, \|u_i^\varepsilon - u_i\|_{L^2([0,T];L^2(\mathbb{R}^2))}^8 \to 0,$$

or equivalently

$$u_i^\varepsilon \to u_i \text{ in } L^{4/3}([0,T]\,;L^{4/3}(\mathbb{R}^2)), i = 1, 2. \tag{3.113}$$

In consequence

$$u_i^\varepsilon A_j \nabla v^\varepsilon \rightharpoonup u_i A_j w \text{ in } L^1([0,T]\,;L^1(\mathbb{R}^2))^2, i, j = 1, 2. \tag{3.114}$$

and

$$\int_0^t \int_{\mathbb{R}^2} \nabla\varphi \cdot (u_i^\varepsilon A_j \nabla v^\varepsilon)\, dx d\tau \to \int_0^t \int_{\mathbb{R}^2} \nabla\varphi \cdot (u_i A_j w)\, dx d\tau,\ \varepsilon \to 0, i, j = 1, 2.$$

Due to (3.109), (3.111) and (3.114), passing to the limit $\varepsilon \to 0$ in (3.110), we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \varphi u_1(x,t) dx - \int_{\mathbb{R}^2} \varphi u_{10}(x) dx \\
&= \mu_1 \int_0^t \int_{\mathbb{R}^2} u_1 \Delta\varphi dx d\tau + \chi_1 \int_0^t \int_{\mathbb{R}^2} \nabla\varphi \cdot (u_1 A_1 w)\, dx d\tau.
\end{aligned} \tag{3.115}$$

Similarly,

$$\begin{aligned}
&\int_{\mathbb{R}^2} \varphi u_2(x,t) dx - \int_{\mathbb{R}^2} \varphi u_{20}(x) dx \\
&= \mu_2 \int_0^t \int_{\mathbb{R}^2} u_1 \Delta\varphi dx d\tau + \chi_2 \int_0^t \int_{\mathbb{R}^2} \nabla\varphi \cdot (u_2 A_2 w)\, dx d\tau.
\end{aligned} \tag{3.116}$$

Now, we claim that $w = \nabla v$ a.e. on $(0,T) \times \mathbb{R}^2$. Indeed, we define

$$n^\varepsilon := a_1 u_1^\varepsilon + a_2 u_2^\varepsilon \text{ and } n := a_1 u_1 + a_2 u_2.$$

Let $\psi \in C_0^\infty((0,T)\times\mathbb{R}^2)$,we have that

$$
\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}^2}\nabla\psi\cdot\left(n^\varepsilon A_j\left(\nabla\mathbf{K}*n^\varepsilon\right)\right)dxd\tau-\int_0^t\int_{\mathbb{R}^2}\nabla\psi\cdot\left(n^\varepsilon A_j\left(\nabla\mathbf{K}^\varepsilon*n^\varepsilon\right)\right)dxd\tau\right|\\
&=\frac{\varepsilon^2}{2\pi}\left|\int_0^t\int_{\mathbb{R}^2\times\mathbb{R}^2}\left(\frac{\nabla\psi(x,t)\cdot A_j(x-y)}{|x-y|^2\left(|x-y|^2+\varepsilon^2\right)}\right)n^\varepsilon(x,t)n^\varepsilon(y,t)dydxd\tau\right|\\
&=\frac{\varepsilon^2}{4\pi}\left|\int_0^t\int_{\mathbb{R}^2\times\mathbb{R}^2}\left(\frac{(\nabla\psi(x,t)-\nabla\psi(y,t))\cdot A_j(x-y)}{|x-y|^2\left(|x-y|^2+\varepsilon^2\right)}\right)n^\varepsilon(x,t)n^\varepsilon(y,t)dydxd\tau\right|\\
&\le\frac{\varepsilon^2}{4\pi}\int_0^t\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{|\nabla\psi(x,t)-\nabla\psi(y,t)|\,|A_j(x-y)|}{|x-y|^2\left(|x-y|^2+\varepsilon^2\right)}n^\varepsilon(x,t)n^\varepsilon(y,t)dydxd\tau\\
&\le\frac{\varepsilon^2C_{25}}{4\pi}\int_0^t\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{|x-y|^2}{|x-y|^2\left(2\varepsilon|x-y|\right)}n^\varepsilon(x,t)n^\varepsilon(y,t)dydxd\tau\\
&=\frac{\varepsilon C_{25}}{8\pi}\int_0^t\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{n^\varepsilon(x,t)n^\varepsilon(y,t)}{|x-y|}dydxd\tau\\
&\le\frac{\varepsilon C_{25}}{8\pi}\int_0^t\left\|1/\left|x\right|*n^\varepsilon\right\|_{L^4(\mathbb{R}^2)}\left\|n^\varepsilon\right\|_{L^{4/3}(\mathbb{R}^2)}d\tau\\
&\le\frac{\varepsilon C_{26}}{8\pi}\int_0^t\left\|n^\varepsilon\right\|^2_{L^{4/3}(\mathbb{R}^2)}d\tau\le\frac{\varepsilon C_{27}}{8\pi}\to0,\varepsilon\to0.
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}^2}\nabla\psi\cdot\left(nA_j\left(\nabla\mathbf{K}*n\right)\right)dxd\tau-\int_0^t\int_{\mathbb{R}^2}\nabla\psi\cdot\left(n^\varepsilon A_j\left(\nabla\mathbf{K}*n^\varepsilon\right)\right)dxd\tau\right|\\
&\le\frac{1}{2\pi}\int_0^t\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{|\nabla\psi(x,t)||A_j(x-y)|}{|x-y|^2}\left|n(x,t)n(y,t)-n^\varepsilon(x,t)n^\varepsilon(y,t)\right|dydxd\tau\\
&\le\frac{C_{28}}{2\pi}\int_0^t\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{\left|n(x,t)n(y,t)-n^\varepsilon(x,t)n^\varepsilon(y,t)\right|}{|x-y|}dydxd\tau\\
&\le\frac{C_{28}}{2\pi}\int_0^t\left(\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{n(x,t)\left|n(y,t)-n^\varepsilon(y,t)\right|}{|x-y|}dydx\right.\\
&\left.+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{\left|n(x,t)-n^\varepsilon(x,t)\right|n^\varepsilon(y,t)}{|x-y|}dydx\right)d\tau\\
&=\frac{C_{28}}{2\pi}\int_0^t\left(\int_{\mathbb{R}^2}\left|n(y,t)-n^\varepsilon(y,t)\right|\int_{\mathbb{R}^2}\frac{n(x,t)}{|y-x|}dxdy\right.\\
&\left.+\int_{\mathbb{R}^2}\left|n(x,t)-n^\varepsilon(x,t)\right|\int_{\mathbb{R}^2}\frac{n^\varepsilon(y,t)}{|x-y|}dydx\right)d\tau\\
&\le\frac{C_{28}}{2\pi}\int_0^t\left(\left\|1/\left|y\right|*n\right\|_{L^4(\mathbb{R}^2)}+\left\|1/\left|x\right|*n^\varepsilon\right\|_{L^4(\mathbb{R}^2)}\right)\left\|n-n^\varepsilon\right\|_{L^{4/3}(\mathbb{R}^2)}d\tau\\
&\le\frac{C_{29}}{2\pi}\int_0^t\left(\left\|n\right\|_{L^{4/3}(\mathbb{R}^2)}+\left\|n^\varepsilon\right\|_{L^{4/3}(\mathbb{R}^2)}\right)\left\|n-n^\varepsilon\right\|_{L^{4/3}(\mathbb{R}^2)}d\tau.
\end{aligned}
$$

The weak semi-continuity of $L^{4/3}(\mathbb{R}^2)$ implies

$$
\|n\|_{L^{4/3}(\mathbb{R}^2)}\le\liminf_{\varepsilon\to0}\|n^\varepsilon\|_{L^{4/3}(\mathbb{R}^2)}\le C_{30}.
$$

Then

$$\left|\int_0^t\int_{\mathbb{R}^2}\nabla\psi\cdot\left(nA_j\left(\nabla\mathbf{K}*n\right)\right)dxd\tau-\int_0^t\int_{\mathbb{R}^2}\nabla\psi\cdot\left(n^\varepsilon A_j\left(\nabla\mathbf{K}*n^\varepsilon\right)\right)dxd\tau\right|$$
$$\leq\frac{C_{31}}{2\pi}\int_0^t\|n-n^\varepsilon\|_{L^{4/3}(\mathbb{R}^2)}d\tau$$
$$\leq\frac{t^{1/4}C_{31}}{2\pi}\left(\int_0^t\|n-n^\varepsilon\|_{L^{4/3}(\mathbb{R}^2)}^{4/3}d\tau\right)^{3/4}\to 0,\varepsilon\to 0.$$

Therefore, we have that

$$\int_0^t\int_{\mathbb{R}^2}\nabla\psi\cdot\left(n^\varepsilon A_j\left(\nabla\mathbf{K}^\varepsilon*n^\varepsilon\right)\right)dxd\tau \tag{3.117}$$
$$\to\int_0^t\int_{\mathbb{R}^2}\nabla\psi\cdot\left(nA_j\left(\nabla\mathbf{K}*n\right)\right)dxd\tau,\varepsilon\to 0.$$

However, by (3.114)

$$\int_0^t\int_{\mathbb{R}^2}\nabla\psi\cdot\left(n^\varepsilon A_j\left(\nabla\mathbf{K}^\varepsilon*n^\varepsilon\right)\right)dxd\tau \tag{3.118}$$
$$\to\int_0^t\int_{\mathbb{R}^2}\nabla\psi\cdot\left(nA_jw\right)dxd\tau,\varepsilon\to 0.$$

By (3.117) and (3.118), we get

$$\int_0^t\int_{\mathbb{R}^2}\nabla\psi\cdot\left(nA_j\left(\nabla v-w\right)\right)dxd\tau=0,\text{for any }\psi\in C_0^\infty((0,T)\times\mathbb{R}^2).$$

Then $nA_j\left(\nabla v-w\right)=C$ a.e. on $(0,T)\times\mathbb{R}^2$. Since $nA_j\left(\nabla v-w\right)\in L^1(\mathbb{R}^2)$ and $n>0$, we get that $\nabla v-w=0$ a.e. on $(0,T)\times\mathbb{R}^2$. This gives the existence of a global weak solution.

Boundedness of the second moment of the weak solution

Lemma 25 *If* $u_{10},u_{20}\in L^1(\mathbb{R}^2,|x|^2dx)$, *then* $|x|^2u_i^\varepsilon,|x|^2u_i\in L^\infty_{loc}(\mathbb{R}^+,L^1(\mathbb{R}^2))$ *for* $i=1,2$.

Proof. Multiplying the first equation of system (3.49) by $|x|^2$ and integrating, we get

$$\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2u_1^\varepsilon(x,t)dx$$
$$=4\mu_1\theta_1+2\chi_1\int_{\mathbb{R}^2}x\cdot\left(u_1^\varepsilon A_1\nabla v^\varepsilon\right)dx$$
$$\leq 4\mu_1\theta_1+2\chi_1\left\||\nabla v^\varepsilon|\right\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2}|x|\,u_1dx$$
$$\leq 4\mu_1\theta_1+\chi_1\left\||\nabla v^\varepsilon|\right\|_{L^\infty(\mathbb{R}^2)}\left(\theta_1+\int_{\mathbb{R}^2}|x|^2u_1^\varepsilon(x,t)dx\right).$$

Summarizing,

$$\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2 u_1^\varepsilon(x,t)dx \le C_{32}\int_{\mathbb{R}^2}|x|^2 u_1^\varepsilon(x,t)dx + C_{33},$$

By Gronwall's inequality (differential form) [39, p. 624], we have that

$$\int_{\mathbb{R}^2}|x|^2 u_1^\varepsilon(x,t)dx \le e^{C_{32}t}\left(\int_{\mathbb{R}^2}|x|^2 u_{10}dx + C_{33}t\right). \tag{3.119}$$

Similarly,

$$\int_{\mathbb{R}^2}|x|^2 u_2^\varepsilon(x,t)dx \le e^{C_{34}t}\left(\int_{\mathbb{R}^2}|x|^2 u_{20}dx + C_{35}t\right). \tag{3.120}$$

Consider a test function $|x|^2\varphi_R(x) \in C_0^\infty(\mathbb{R}^2)$, where $\varphi_R(x)$ be defined as in (3.72), which grows to $|x|^2$ as $R\to\infty$, letting $\varphi = |x|^2\varphi_R(x)$ in (3.115) we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2}|x|^2\varphi_R u_1(x,t)dx - \int_{\mathbb{R}^2}|x|^2\varphi_R u_{10}dx\\&= \mu_1\int_0^t\int_{\mathbb{R}^2}u_1\Delta\left(|x|^2\varphi_R\right)dxd\tau + \chi_1\int_0^t\int_{\mathbb{R}^2}\nabla\left(|x|^2\varphi_R\right)\cdot\left(u_1A_1\nabla v\right)dxd\tau.\end{aligned}$$

Notice that

$$\begin{aligned}\left|\Delta(|x|^2\varphi_R)\right| &\le 4\varphi_R + 4|x|\,|\nabla\varphi_R| + |x|^2|\Delta\varphi_R|\\&\le 4 + 4(2R)\frac{C^*}{R} + (2R)^2\frac{C^*}{R^2} = 4 + 12C^*,\end{aligned}$$

and

$$\begin{aligned}\left|\nabla\left(|x|^2\varphi_R\right)\right| &\le 2|x|\varphi_R + |x|^2|\nabla\varphi_R|\\&\le 2|x| + |x|(2R)\frac{C^*}{R} = 2(1+C^*)|x|.\end{aligned}$$

Therefore, we obtain

$$\left|\int_0^t\int_{\mathbb{R}^2}u_1\Delta\left(|x|^2\varphi_R\right)dxd\tau\right| \le (4+12C^*)\,\theta_1 T,$$

and

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}^2}\nabla\left(|x|^2\varphi_R\right)\cdot\left(u_1A_1\nabla v\right)dxd\tau\right| \le 2(1+C^*)\int_0^t\int_{\mathbb{R}^2}|x|\,u_1\,|A_1\nabla v|\,dxd\tau\\&\le 2\left(1+C^*\right)\||\nabla v|\|_{L^\infty(\mathbb{R}^2)}\int_0^t\int_{\mathbb{R}^2}|x|\,u_1dxd\tau\\&\le \left(1+C^*\right)\||\nabla v|\|_{L^\infty(\mathbb{R}^2)}\left(\theta_1T + \int_0^t\int_{\mathbb{R}^2}|x|^2u_1(x,t)dxd\tau\right)\\&\le \tfrac{(1+C^*)\|a_1u_1+a_2u_2\|^{2/3}_{L^4(\mathbb{R}^2)}(a_1\theta_1+a_2\theta_2)^{1/3}C_{36}}{2\pi}\left(\theta_1T + \int_0^t\int_{\mathbb{R}^2}|x|^2u_1(x,t)dxd\tau\right)\\&\le \tfrac{(\max\{a_1,a_2\})^{2/3}(1+C^*)(a_1\theta_1+a_2\theta_2)^{1/3}C_{37}}{2\pi}\left(\theta_1T + \int_0^t\int_{\mathbb{R}^2}|x|^2u_1(x,t)dxd\tau\right).\end{aligned}$$

where the last lines is given by the inequality (3.21) and the semi-continuity of

$$\|u_i\|_{L^4(\mathbb{R}^2)} \leq \liminf_{\varepsilon\to 0} \|u_i^\varepsilon\|_{L^4(\mathbb{R}^2)} \leq C_{38}.$$

As $R \to \infty$ we find that

$$\int_{\mathbb{R}^2} |x|^2 u_1(x,t)dx \leq C_{39} \int_0^t \int_{\mathbb{R}^2} |x|^2 u_1(x,t)dxd\tau + \int_{\mathbb{R}^2} |x|^2 u_{10}(x)dx + C_{40}T.$$

By Gronwall's inequality (integral form) [39, p. 625], we have that

$$\int_{\mathbb{R}^2} |x|^2 u_1(x,t)dx \leq \left(\int_{\mathbb{R}^2} |x|^2 u_{10}dx + C_{40}T\right)\left(1 + C_{39}te^{C_{39}t}\right). \tag{3.121}$$

Similarly,

$$\int_{\mathbb{R}^2} |x|^2 u_2(x,t)dx \leq \left(\int_{\mathbb{R}^2} |x|^2 u_{20}dx + C_{42}T\right)\left(1 + C_{41}te^{C_{41}t}\right). \tag{3.122}$$

■

The energy inequality of the weak solution Integrating (3.66) in time from 0 to t follows

$$\begin{aligned}
&\frac{\mu_1 a_1}{\chi_1 \cos\alpha_1}\int_{\mathbb{R}^2} u_1^\varepsilon \ln u_1^\varepsilon dx + \frac{\mu_2 a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2} u_2^\varepsilon \ln u_2^\varepsilon dx \\
&-\frac{a_1}{2}\int_{\mathbb{R}^2} u_1^\varepsilon v^\varepsilon dx - \frac{a_2}{2}\int_{\mathbb{R}^2} u_2^\varepsilon v^\varepsilon dx \\
&+\frac{a_1}{\chi_1\cos\alpha_1}\int_0^t\int_{\mathbb{R}^2} u_1^\varepsilon \left|\nabla(\mu_1 \ln u_1^\varepsilon - \chi_1\cos\alpha_1 v^\varepsilon)\right|^2 dxdt \\
&+\frac{a_2}{\chi_2\cos\alpha_2}\int_0^t\int_{\mathbb{R}^2} u_2^\varepsilon \left|\nabla(\mu_2 \ln u_2^\varepsilon - \chi_2\cos\alpha_2 v^\varepsilon)\right|^2 dxdt \\
&= E(0).
\end{aligned} \tag{3.123}$$

The goal of this last part is to take the limit $\varepsilon \to 0$ in (3.123) to derive the energy dissipation (3.9). For clarity, we divide this proof into three steps.

Step 1: Pass to the limit in the entropy functionals.

We claim that

$$\int_{\mathbb{R}^2} u_i^\varepsilon \ln u_i^\varepsilon dx \to \int_{\mathbb{R}^2} u_i \ln u_i dx \text{ a.e.on } [0,T]\,, \text{ for } i = 1,2. \tag{3.124}$$

Indeed, by (3.60) we have that

$$\begin{aligned}
u_i^\varepsilon |\ln u_i^\varepsilon| &= u_i^\varepsilon \ln^+ u_i^\varepsilon + u_i^\varepsilon \ln^- u_i^\varepsilon \\
&\leq |u_i^\varepsilon|^2 + \left(1+|x|^2\right)^{-2} + 4u_i^\varepsilon \ln(1+|x|^2) =: h_i^\varepsilon,
\end{aligned}$$

for $i = 1,2$.Assuming that $u_{10}, u_{20} \in L^1(\mathbb{R}^2, |x|^2 dx)$, it can be shown that (See Lemma 25)

$$\int_{\mathbb{R}^2} u_i^\varepsilon |x|^2 dx, \int_{\mathbb{R}^2} u_i |x|^2 dx \leq C_{43} < +\infty. \tag{3.125}$$

Notice that the weak convergence of u_i^ε to u_i in $L^1(\mathbb{R}^2)$ for a.e. on $[0,T]$ and the assumption (3.125) are enough to prove that

$$\int_{\mathbb{R}^2} u_i^\varepsilon \ln(1+|x|^2)dx \to \int_{\mathbb{R}^2} u_i \ln(1+|x|^2)dx \text{ a.e.on } [0,T]\,, \text{ for } i=1,2. \quad (3.126)$$

In fact, for any $R>1$, we have that

$$\begin{aligned}&\left|\int_{\mathbb{R}^2} u_i^\varepsilon \ln(1+|x|^2)dx - \int_{\mathbb{R}^2} u_i \ln(1+|x|^2)dx\right| \\ &\le \left|\int_{|x|\le R} (u_i^\varepsilon - u_i)\ln(1+|x|^2)dx\right| + \int_{|x|>R} (u_i^\varepsilon + u_i)\ln(1+|x|^2)dx.\end{aligned}$$

On the one hand, by (3.125) we obtain

$$\begin{aligned}\int_{|x|>R} (u_i^\varepsilon + u_i)\ln(1+|x|^2)dx &\le \int_{|x|>R} (u_i^\varepsilon + u_i)\,|x|\,dx \\ &\le \frac{1}{R}\int_{\mathbb{R}^2} (u_i^\varepsilon + u_i)\,|x|^2\,dx \\ &\le \frac{2C_{43}}{R} \to 0, R\to\infty.\end{aligned}$$

On the other hand, by weak convergence $u_i^\varepsilon \rightharpoonup u_i$ in $L^1(\mathbb{R}^2)$

$$\left|\int_{\mathbb{R}^2} (u_i^\varepsilon - u_i)\ln(1+|x|^2)1_{\{|x|\le R\}}dx\right| \to 0, \varepsilon\to 0.$$

Therefore

$$\left|\int_{\mathbb{R}^2} \ln(1+|x|^2)u_1^\varepsilon(x,t)dx - \int_{\mathbb{R}^2} \ln(1+|x|^2)u_1(x,t)dx\right| \to 0, \varepsilon\to 0.$$

Using the strong convergence of u_i^ε to u_i in $L^2(\mathbb{R}^2)$ for a.e. on $[0,T]$ and (3.126), we get

$$h_i^\varepsilon \to h_i \text{ a.e. on } R^2\times[0,T] \text{ and } \int_{\mathbb{R}^2} h_i^\varepsilon dx \to \int_{\mathbb{R}^2} h_i dx < \infty,$$

where the function $h_i := |u_i|^2 + \left(1+|x|^2\right)^{-2} + 4u_i\ln(1+|x|^2)$. Therefore, we can apply the General Lebesgue Dominated Convergence Theorem (See [75, Theorem 19 p. 89]) to conclude (3.124).

Step 2: Pass to the limit in the free energy dissipation.

The lower semi-continuity of the energy dissipation is followed from the Lemma 22 items (**i**) and (**vii**) that

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^2} |2\mu_i\nabla\sqrt{u_i} - \chi_i\cos\alpha_i\sqrt{u_i}\nabla v)|^2\,dxdt \qquad (3.127)\\ &\le \liminf_{\varepsilon\to 0}\int_0^t\int_{\mathbb{R}^2}\left|2\mu_i\nabla\sqrt{u_i^\varepsilon} - \chi_i\cos\alpha_i\sqrt{u_i^\varepsilon}\nabla v^\varepsilon)\right|^2 dxdt, i=1,2.\end{aligned}$$

In fact, the weak convergence of $\sqrt{u_i^\varepsilon}$ towards $\sqrt{u_i}$ in $L^2([0,T]\,;H^1(\mathbb{R}^2))$ holds due to its uniform boundedness given by inequality (3.102) and the mass conservation property, where the limit is identified by using the a.e. convergence of u_i^ε from the strong convergence in $L^2([0,T]\,;L^2(\mathbb{R}^2))$. Moreover, notice that

$$\begin{aligned}
&\left\|\sqrt{u_i^\varepsilon}-\sqrt{u_i}\right\|^2_{L^2([0,T];L^2(\mathbb{R}^2))}\\
&=\int_0^t\int_{\mathbb{R}^2}(u_i^\varepsilon-u_i)\,dxdt+2\int_0^t\int_{\mathbb{R}^2}\sqrt{u_i}\left(\sqrt{u_i}-\sqrt{u_i^\varepsilon}\right)dxdt\\
&=2\int_0^t\int_{\mathbb{R}^2}\sqrt{u_i}\left(\sqrt{u_i}-\sqrt{u_i^\varepsilon}\right)dxdt\to 0,
\end{aligned}$$

since $\sqrt{u_i}\in L^2([0,T]\,;L^2(\mathbb{R}^2))$ and $\sqrt{u_i^\varepsilon}\rightharpoonup\sqrt{u_i}$ in $L^2([0,T]\,;L^2(\mathbb{R}^2))$. Then, we have that $\sqrt{u_i^\varepsilon}\to\sqrt{u_i}$ in $L^2([0,T]\,;L^2(\mathbb{R}^2))$. On the other hand, due the inequality (3.106), we have the weak convergence of $\sqrt{u_i^\varepsilon}\nabla v^\varepsilon\rightharpoonup\widetilde{w}$ in $L^2([0,T]\,;L^2(\mathbb{R}^2))^2$. In order to identify the limit we use the strong convergence $u_i^\varepsilon\to u_i$ in $L^{4/3}([0,T]\,;L^{4/3}(\mathbb{R}^2))$ and the weak convergence of $\nabla v^\varepsilon\rightharpoonup\nabla v$ in $L^4([0,T];L^4(\mathbb{R}^2))^2$, then, we obtain that $u_i^\varepsilon\nabla v^\varepsilon\rightharpoonup u_i\nabla v=\sqrt{u_i}\sqrt{u_i}\nabla v$ in $L^1([0,T];L^1(\mathbb{R}^2))^2$. On the other hand, due to the strong convergence $\sqrt{u_i^\varepsilon}\to\sqrt{u_i}$ in $L^2([0,T];L^2(\mathbb{R}^2))$ and the weak convergence of $\sqrt{u_i^\varepsilon}\nabla v^\varepsilon\rightharpoonup\widetilde{w}$ in $L^2([0,T]\,;L^2(\mathbb{R}^2))^2$, we also have that $u_i^\varepsilon\nabla v^\varepsilon=\sqrt{u_i^\varepsilon}\sqrt{u_i^\varepsilon}\nabla v^\varepsilon\rightharpoonup\sqrt{u_i}\widetilde{w}$ in $L^1([0,T]\,;L^1(\mathbb{R}^2))^2$. Then, by uniqueness of the weak limit $\sqrt{u_i}\left(\sqrt{u_i}\nabla v-\widetilde{w}\right)=0$ a.e. on $(0,T)\times\mathbb{R}^2$. Since $u_i>0$, we get that $\sqrt{u_i}\nabla v-\widetilde{w}=0$ a.e. on $(0,T)\times\mathbb{R}^2$.

Step 3: Pass to the limit in the potential energy functionals.
We need to verify that

$$\int_{\mathbb{R}^2}n^\varepsilon v^\varepsilon dx\to\int_{\mathbb{R}^2}nvdx\ \text{a.e.on}\ [0,T]\,,\ \text{for } i=1,2,\tag{3.128}$$

where $n^\varepsilon:=a_1u_1^\varepsilon+a_2u_2^\varepsilon$ and $n:=a_1u_1+a_2u_2$. We rewrite

$$\begin{aligned}
&-4\pi\int_{\mathbb{R}^2}(n^\varepsilon v^\varepsilon-nv)\,dx\\
&=\int\int_{|x-y|<1}\ln\frac{|x-y|^2+\varepsilon^2}{|x-y|^2}n^\varepsilon(x,t)n^\varepsilon(y,t)dydx\\
&+2\int\int_{|x-y|<1}(n^\varepsilon(y,t)-n(y,t))\,n^\varepsilon(x,t)\ln|x-y|dydx\\
&+2\int\int_{|x-y|<1}(n^\varepsilon(x,t)-n(x,t))\,n(y,t)\ln|x-y|dydx\\
&+\int\int_{|x-y|\geq1}\ln\frac{|x-y|^2+\varepsilon^2}{|x-y|^2}n^\varepsilon(x,t)n^\varepsilon(y,t)dydx\\
&+2\int\int_{|x-y|\geq1}(n^\varepsilon(y,t)-n(y,t))\,n^\varepsilon(x,t)\ln|x-y|dydx\\
&+2\int\int_{|x-y|\geq1}(n^\varepsilon(x,t)-n(x,t))\,n(y,t)\ln|x-y|dydx\\
&=:\sum_{j=1}^{6}T_j.
\end{aligned}$$

By the Young's convolution inequality (3.91), we have that

$$\begin{aligned}
&T_1+T_2+T_3\\
&\le \|n^\varepsilon\|_{L^2(\mathbb{R}^2)}\|n^\varepsilon\|_{L^2(\mathbb{R}^2)}\int_{\mathbb{R}^2}1_{\{|x|<1\}}\ln\frac{|x|^2+\varepsilon^2}{|x|^2}dx\\
&+2\|n^\varepsilon-n\|_{L^2(\mathbb{R}^2)}\left(\|n^\varepsilon\|_{L^2(\mathbb{R}^2)}+\|n\|_{L^2(\mathbb{R}^2)}\right)\int_{\mathbb{R}^2}1_{\{|x|<1\}}\ln\frac{1}{|x|}dx\\
&=2\pi\|n^\varepsilon\|_{L^2(\mathbb{R}^2)}\|n^\varepsilon\|_{L^2(\mathbb{R}^2)}\int_0^1 r\ln\frac{r^2+\varepsilon^2}{r^2}dr\\
&+4\pi\|n^\varepsilon-n\|_{L^2(\mathbb{R}^2)}\left(\|n^\varepsilon\|_{L^2(\mathbb{R}^2)}+\|n\|_{L^2(\mathbb{R}^2)}\right)\int_0^1 r\ln\frac{1}{r}dr\\
&=2\pi\|n^\varepsilon\|_{L^2(\mathbb{R}^2)}\|n^\varepsilon\|_{L^2(\mathbb{R}^2)}\int_0^1 r\ln\frac{r^2+\varepsilon^2}{r^2}dr\\
&+\pi\|n^\varepsilon-n\|_{L^2(\mathbb{R}^2)}\left(\|n^\varepsilon\|_{L^2(\mathbb{R}^2)}+\|n\|_{L^2(\mathbb{R}^2)}\right).
\end{aligned}$$

Notice that

$$2\int_0^1 r\ln\frac{r^2+\varepsilon^2}{r^2}dr=(1+\varepsilon^2)\ln(1+\varepsilon^2)-\varepsilon^2\ln\varepsilon^2\to 0,\ \text{as } \varepsilon\to 0.$$

Therefore, for any $0<t<T$

$$T_1+T_2+T_3\to 0,\ \text{as } \varepsilon\to 0.$$

For T_4, we have that

$$\begin{aligned}
T_4&\le \ln\left(1+\varepsilon^2\right)\int\int_{|x-y|\ge 1}n^\varepsilon(x,t)n^\varepsilon(y,t)dydx\\
&\le \ln\left(1+\varepsilon^2\right)(a_1\theta_1+a_2\theta_2)^2\to 0,\ \text{as } \varepsilon\to 0.
\end{aligned}$$

Next, we compute T_3 or T_5 using the Cauchy-Schwarz inequality in the following way

$$
\begin{aligned}
&(T_3)^2 \\
&= \left(2\int_{\mathbb{R}^2} |n^\varepsilon(y,t)-n(y,t)| \left(\int_{|x-y|\geq 1} n^\varepsilon(x,t)\ln|x-y|dx\right)dy\right)^2 \\
&\leq 4\,\|n^\varepsilon - n\|_{L^1(\mathbb{R}^2)} \int_{\mathbb{R}^2} |n^\varepsilon(y,t)-n(y,t)| \left|\int_{|x-y|\geq 1} n^\varepsilon(x,t)\ln|x-y|dx\right|^2 dy \\
&\leq 4(a_1\theta_1+a_2\theta_2)\,\|n^\varepsilon - n\|_{L^1(\mathbb{R}^2)} \int_{\mathbb{R}^2} |n^\varepsilon(y,t)-n(y,t)| \\
&\int_{|x-y|\geq 1} n^\varepsilon(x,t)\left(\ln|x-y|\right)^2 dxdy \\
&\leq 4(a_1\theta_1+a_2\theta_2)\,\|n^\varepsilon - n\|_{L^1(\mathbb{R}^2)} \\
&\int_{\mathbb{R}^2}\int_{|x-y|\geq 1} \left(n^\varepsilon(y,t)+n(y,t)\right) n^\varepsilon(x,t)|x-y|^2 dxdy \\
&\leq 8(a_1\theta_1+a_2\theta_2)\,\|n^\varepsilon - n\|_{L^1(\mathbb{R}^2)} \\
&\int_{\mathbb{R}^2}\int_{|x-y|\geq 1} \left(n^\varepsilon(y,t)+n(y,t)\right) n^\varepsilon(x,t)\left(|x|^2+|y|^2\right) dxdy \\
&\leq 8(a_1\theta_1+a_2\theta_2)^2\,\|n^\varepsilon - n\|_{L^1(\mathbb{R}^2)} \left(3\int_{\mathbb{R}^2} n^\varepsilon(x,t)|x|^2dx + \int_{\mathbb{R}^2} n(x,t)|x|^2dx\right) \\
&\to 0 \text{ as } \varepsilon \to 0.
\end{aligned}
$$

Finally, combining (3.126), (3.127) and (3.128), letting $\varepsilon \to 0$ in (3.123), we obtain

$$
\begin{aligned}
&\frac{\mu_1 a_1}{\chi_1\cos\alpha_1}\int_{\mathbb{R}^2} u_1\ln u_1 dx + \frac{\mu_2 a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2} u_2\ln u_2 dx \\
&-\frac{a_1}{2}\int_{\mathbb{R}^2} u_1 v dx - \frac{a_2}{2}\int_{\mathbb{R}^2} u_2 v dx \\
&+\frac{a_1}{\chi_1\cos\alpha_1}\int_0^t\int_{\mathbb{R}^2} u_1\left|\nabla(\mu_1\ln u_1 - \chi_1\cos\alpha_1 v)\right|^2 dxdt \\
&+\frac{a_2}{\chi_2\cos\alpha_2}\int_0^t\int_{\mathbb{R}^2} u_2\left|\nabla(\mu_2\ln u_2 - \chi_2\cos\alpha_2 v)\right|^2 dxdt \\
&\leq E(0).
\end{aligned}
$$

3.2.2 Case $\alpha_1, \alpha_2 \in \left(-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]$

Theorem 26 *Assume that a_1, a_2 are non-negative constants and If $\alpha_1, \alpha_2 \in \left(-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]$, i.e., both species move away of the gradient of chemical concentration, then for any initial masses θ_i, $i = 1, 2$, the system (3.1) has a global weak solution satisfying the energy dissipation (3.9) under the additional hypothesis $u_{10}\,|x|^2, u_{20}\,|x|^2 \in L^1(\mathbb{R}^2)$.*

In this case, it is not necessary the use of any energy functional and instead, a direct approach is enough to bound the L^p-norms in time. Indeed, if we proceed as in the first part of the proof of Lemma 22 item **(i)**, we arrive at the following estimate

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2}|u_1^\varepsilon|^p\,dx\\&=-\frac{4(p-1)}{p}\mu_1\int_{\mathbb{R}^2}\left|\nabla\left(u_1^\varepsilon\right)^{p/2}\right|^2dx\\&+(p-1)\chi_1\cos\alpha_1\int_{\mathbb{R}^2}\underbrace{(u_1^\varepsilon)^p\left(-\Delta v^\varepsilon\right)}_{>0}dx.\end{aligned}$$

Since for all $\alpha_1\in\left(-\pi,-\frac{\pi}{2}\right]\cup\left[\frac{\pi}{2},\pi\right]$, it holds that $\cos\alpha_1\le 0$, then it follows that $\int_{\mathbb{R}^2}|u_1^\varepsilon|^p\,dx$ is non-increasing and thus bounded from above due to the assumption $u_{10}\in L^p(\mathbb{R}^2)$. Similarly, $\int_{\mathbb{R}^2}|u_2^\varepsilon|^p\,dx$ is also bounded from above. Therefore, we have that for any initial masses $\theta_i, i=1,2$, the solution $(u_1^\varepsilon,u_2^\varepsilon)$ of (3.49) is bounded in $L^\infty_{loc}(\mathbb{R}^+,L^p(\mathbb{R}^2))$, for all $1<p<\infty$. Hence, under assumption (3.6), we can pass to the limit $\varepsilon\to 0$ by applying the same argument described in the previous case, which allow us to conclude the global existence of weak solution for system (3.1) satisfying the energy dissipation (3.9) under the additional hypothesis $u_{10}\,|x|^2, u_{20}\,|x|^2\in L^1(\mathbb{R}^2)$.

3.3 Finite time blow-up for radially symmetric solutions

The purpose in this section is to derive sharp conditions on the initial masses for having blow-up for system (3.1). We adapt the ideas for the multi-species case introduced in [28]. The challenge of this adaptation remains on the lack of symmetry for the anti-gradient operator $\nabla^\perp$. In light of this, we will only consider radial initial conditions u_{10},u_{20}, leaving the question for the nonradial case open. We also show that even when the total moment increases the blow-up is possible.

The next lemma constitutes a key remark for proving the possibility of having blow-up for radially symmetric solutions. It constitutes a generalization of a Lemma in [45, p. 46].

Lemma 27 *Let* $\mathbf{K}$ *be the fundamental solution of the Laplace operator defined as*

$$\mathbf{K}(x):=-\frac{1}{2\pi}\ln|x|\,,\ x\in\mathbb{R}^2, x\neq 0.$$

Let ρ_1 *and* ρ_2 *smooth real-valued functions on* $\mathbb{R}^2$ *that are radially symmetric, i.e., they depend only on the length* $|x|$ *or in other words, they are invariant under all rotations centered at the origin. Assume that* $K*\rho_1$ *is defined as a* C^1*-function on* $\mathbb{R}^2$*. Then*

$$\nabla^\perp(\mathbf{K}*\rho_1)\cdot\nabla\rho_2=0.$$

Proof. Note that, since ρ_1 and E are radially symmetric, $\rho_1(\mathbf{R}x)=\rho_1(x)$ and $\mathbf{K}(\mathbf{R}x)=\mathbf{K}(x)$ for any rotation matrix $\mathbf{R}\in M_{2x2}(\mathbb{R})$, i.e.,

$$\mathbf{R}=\begin{pmatrix}\cos\alpha_1 & -\sin\alpha_1\\ \sin\alpha_1 & \cos\alpha_1\end{pmatrix}.$$

Moreover, we have that

$$(\mathbf{K} * \rho_1)(\mathbf{R}x) = \int_{\mathbb{R}^2} \mathbf{K}(\mathbf{R}x - y)\rho_1(y)dy.$$

Taking $z = \mathbf{R}^{-1}y$, where

$$\mathbf{R}^{-1} = \mathbf{R}^T = \begin{pmatrix} \cos\alpha_1 & \sin\alpha_1 \\ -\sin\alpha_1 & \cos\alpha_1 \end{pmatrix}.$$

Note that, since $\det(R) = 1$, we have that $dy = dz$.We obtain,

$$\begin{aligned}(\mathbf{K} * \rho_1)(\mathbf{R}x) &= \int_{\mathbb{R}^2} \mathbf{K}(\mathbf{R}x - \mathbf{R}z)\rho_1(\mathbf{R}z)dz \\ &= \int_{\mathbb{R}^2} \mathbf{K}(\mathbf{R}(x - z))\rho_1(\mathbf{R}z)dz \\ &= \int_{\mathbb{R}^2} \mathbf{K}(x - z)\rho_1(z)dz = (\mathbf{K} * \rho_1)(x).\end{aligned}$$

So, we have that $\mathbf{K} * \rho_1$ is also radially symmetric, i.e., they depend only on the length $|x|$. Note that if f is any radially symmetric function then

$$\nabla f = \frac{df}{d|x|}\frac{d|x|}{dx} = \frac{df}{d|x|}\frac{x}{|x|}.$$

Hence, both gradients $\nabla(\mathbf{K} * \rho_1)$ and $\nabla\rho_2$ are parallel to the vector $x/|x|$ for $x \neq 0$. In particular, $\nabla(\mathbf{K} * \rho_1)$ is parallel to $\nabla\rho_2$ for $x \neq 0$. On the other hand, if $g \in C^1(\mathbb{R}^2), \nabla^{\perp}g$ is orthogonal to ∇g. So, $\nabla^{\perp}(\mathbf{K} * \rho_1)$ is orthogonal to $\nabla(\mathbf{K} * \rho_1)$. Therefore $\nabla^{\perp}(\mathbf{K} * \rho_1)$ is orthogonal to $\nabla\rho_2$, i.e.

$$\nabla^{\perp}(\mathbf{K} * \rho_1) \cdot \nabla\rho_2 = 0.$$

■

Proof of theorem 4. Let us first assume that θ_1 and θ_2 satisfy (3.13) and $a_1\cos\alpha_1, a_2\cos\alpha_2 > 0$. We start by decomposing the matrix A_1 into the form

$$A_1 = \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ \sin\alpha_1 & \cos\alpha_1 \end{pmatrix} = \cos\alpha_1 I + \sin\alpha_1 R, \tag{3.129}$$

where I denotes the identity matrix and

$$R := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Next, we re-write the equation for u_1 in the form

$$\begin{aligned}&\partial_t u_1 \\ &= \mu_1\triangle u_1 - \chi_1\cos\alpha_1\nabla\cdot(u_1\nabla v) - \chi_1\sin\alpha_1\nabla\cdot(u_1R\nabla v) \\ &= \mu_1\triangle u_1 - \chi_1\cos\alpha_1\nabla\cdot(u_1\nabla v) - \chi_1\sin\alpha_1\nabla\cdot(u_1\nabla^{\perp}v) \\ &= \mu_1\triangle u_1 - \chi_1\cos\alpha_1\nabla\cdot(u_1\nabla v) - \chi_1\sin\alpha_1(\nabla u_1\cdot\nabla^{\perp}v + u_1\underbrace{\nabla\cdot\nabla^{\perp}v}_{=0}). \\ &= \mu_1\triangle u_1 - \chi_1\cos\alpha_1\nabla\cdot(u_1\nabla v) - \chi_1\sin\alpha_1(\nabla u_1\cdot\nabla^{\perp}v).\end{aligned}$$

After Lemma 27, the last term vanishes, thus

$$\partial_t u_1 = \mu_1 \triangle u_1 - \chi_1 \cos\alpha_1 \nabla\cdot(u_1\nabla v). \tag{3.130}$$

Multiplying (3.130) by $|x|^2$ and integrating, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2 u_1 dx &= \mu_1\int_{\mathbb{R}^2}|x|^2 \triangle u_1 dx - \chi_1\cos\alpha_1\int_{\mathbb{R}^2}|x|^2\nabla\cdot(u_1\nabla v)dx\\ &= 4\mu_1\theta_1 + 2\chi_1\cos\alpha_1\int_{\mathbb{R}^2}x\cdot(u_1\nabla v)dx. \end{aligned} \tag{3.131}$$

Using the representation of ∇v, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2 u_1 dx \qquad (3.132)\\ &= 4\mu_1\theta_1 - \frac{\chi_1\cos\alpha_1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} x\cdot\frac{x-y}{|x-y|^2}u_1(x,t)(a_1u_1(y,t)+a_2u_2(y,t))dydx.\end{aligned}$$

Similarly

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2 u_2 dx \qquad (3.133)\\ &= 4\mu_2\theta_2 - \frac{\chi_2\cos\alpha_2}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} x\cdot\frac{x-y}{|x-y|^2}u_2(x,t)(a_1u_1(y,t)+a_2u_2(y,t))dydx.\end{aligned}$$

The expression $\frac{2\pi a_1}{\chi_1\cos\alpha_1}$(3.132)+$\frac{2\pi a_2}{\chi_2\cos\alpha_2}$(3.133) gives

$$\begin{aligned}&\frac{d}{dt}\left(\frac{2\pi a_1}{\chi_1\cos\alpha_1}\int_{\mathbb{R}^2}|x|^2u_1dx+\frac{2\pi a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2}|x|^2u_2dx\right)\\ &=\frac{8\pi\mu_1a_1}{\chi_1\cos\alpha_1}\theta_1+\frac{8\pi\mu_2a_2}{\chi_2\cos\alpha_2}\theta_2\\ &-2\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{x\cdot(x-y)}{|x-y|^2}\left(a_1u_1(x,t)+a_2u_2(x,t)\right)(a_1u_1(y,t)+a_2u_2(y,t))dydx.\end{aligned}$$

The symmetry in the variables x and y in the last integral implies

$$\begin{aligned}&\frac{d}{dt}\left(\frac{2\pi a_1}{\chi_1\cos\alpha_1}\int_{\mathbb{R}^2}|x|^2u_1dx+\frac{2\pi a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2}|x|^2u_2dx\right)\\ &=\frac{8\pi\mu_1a_1}{\chi_1\cos\alpha_1}\theta_1+\frac{8\pi\mu_2a_2}{\chi_2\cos\alpha_2}\theta_2\\ &-\int_{\mathbb{R}^2\times\mathbb{R}^2}\tfrac{(x-y)\cdot(x-y)}{|x-y|^2}\left(a_1u_1(x,t)+a_2u_2(x,t)\right)(a_1u_1(y,t)+a_2u_2(y,t))dydx\\ &=\frac{8\pi\mu_1a_1}{\chi_1\cos\alpha_1}\theta_1+\frac{8\pi\mu_2a_2}{\chi_2\cos\alpha_2}\theta_2-(a_1\theta_1+a_2\theta_2)^2.\end{aligned}$$

Let the second moment $m(t)$ with respect to the origin for the whole population, defined by

$$m(t):=\frac{2\pi a_1}{\chi_1\cos\alpha_1}\int_{\mathbb{R}^2}|x|^2u_1dx+\frac{2\pi a_2}{\chi_2\cos\alpha_2}\int_{\mathbb{R}^2}|x|^2u_2dx.$$

Thus

$$\frac{d}{dt}m(t) = \frac{8\pi\mu_1 a_1}{\chi_1\cos\alpha_1}\theta_1 + \frac{8\pi\mu_2 a_2}{\chi_2\cos\alpha_2}\theta_2 - (a_1\theta_1 + a_2\theta_2)^2.$$

Integrating on $(0,t)$, we obtain that

$$m(t) = m(0) + \left(\frac{8\pi\mu_1 a_1}{\chi_1\cos\alpha_1}\theta_1 + \frac{8\pi\mu_2 a_2}{\chi_2\cos\alpha_2}\theta_2 - (a_1\theta_1 + a_2\theta_2)^2\right)t. \tag{3.134}$$

The inequality (3.11) implies now that $m(t)$ should become negative in finite time which is impossible since u_1 and u_2 are non-negative and $a_1\cos\alpha_1, a_2\cos\alpha_2 > 0$. In conclusion $T_{\max} < \infty$.

We proceed now to show that the inequalities $a_1 a_2 \geq 0$, $a_i\cos\alpha_i > 0$ for at least one index $i \in \{1,2\}$ and the condition (3.12) implies $T_{\max} < \infty$. In this case, we defined the second moment $m_i(t)$ with respect to the origin for each variable

$$m_i(t) := \int_{\mathbb{R}^2} |x|^2\, u_i(x,t)dx.$$

as well as the cumulative mass $M_i(r,t)$

$$M_i(r,t) := \int_{B(0,r)} u_i(x,t)dx = 2\pi\int_0^r u_i(\rho,t)\rho d\rho.$$

By (3.131) we have that

$$\frac{d}{dt}m_i(t) = 4\mu_i\theta_i + 2\chi_i\cos\alpha_i\int_{\mathbb{R}^2} x\cdot(u_i\nabla v)dx. \tag{3.135}$$

In polar coordinates

$$-\triangle v = -\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = a_1u_1 + a_2u_2.$$

Thus

$$\begin{aligned} r\frac{dv}{dr} &= -a_1\int_0^r u_1(\rho,t)\rho d\rho - a_2\int_0^r u_2(\rho,t)\rho d\rho \\ &= -\frac{a_1}{2\pi}\int_{B(0,r)} u_1(x,t)dx - \frac{a_2}{2\pi}\int_{B(0,r)} u_2(x,t)dx \\ &= -\frac{a_1}{2\pi}M_1(r,t) - \frac{a_2}{2\pi}M_2(r,t). \end{aligned} \tag{3.136}$$

Moreover $x\cdot\nabla f = r\frac{df}{dr}$, then

$$\cos\alpha_i\int_{\mathbb{R}^2} x\cdot(u_i\nabla v)dx = 2\pi\cos\alpha_i\int_0^{+\infty} u_i\rho\frac{dv}{d\rho}\rho d\rho. \tag{3.137}$$

Replacing (3.136) in (3.137), we obtain

$$\begin{aligned} &\cos\alpha_i\int_{\mathbb{R}^2} x\cdot(u_i\nabla v)dx \\ &= -\cos\alpha_i\int_0^{+\infty}(a_1M_1(\rho,t)u_i(\rho,t) + a_2M_2(\rho,t)u_i(\rho,t))\rho d\rho \\ &< -a_i\cos\alpha_i\int_0^{+\infty}(M_i(\rho,t)u_i(\rho,t)\rho d\rho \\ &= -\frac{a_i}{2\pi}\int_0^{+\infty} M_i\frac{dM_i}{d\rho}d\rho = -\frac{a_i}{4\pi}\int_0^{+\infty}\frac{d}{d\rho}M_i^2 d\rho = -\frac{a_i}{4\pi}\theta_i^2, \end{aligned} \tag{3.138}$$

since $a_j \cos\alpha_i \geq 0$. Replacing (3.138) in (3.135), we get

$$\begin{aligned}\frac{d}{dt}m_i(t) &< 4\mu_i\theta_i - \frac{\chi_i a_i \cos\alpha_i}{2\pi}\theta_i^2 \\ &= 4\mu_i\theta_i\left(1 - \frac{a_i\chi_i\cos\alpha_i}{8\pi\mu_i}\theta_i\right).\end{aligned} \tag{3.139}$$

Therefore, we have $T_{\max} < \infty$ when

$$\theta_i > \frac{8\pi\mu_i}{a_i\chi_i\cos\alpha_i}.$$

For the sake of simplicity, we have just performed a formal proof. However, this argument can be made rigorous by taking in the weak formulation the test function $|x|^2\varphi_R(x) \in C_0^\infty(\mathbb{R}^2)$, where $\varphi_R(x)$ is defined as in (3.72), which grows to $|x|^2$ as $R \to \infty$. Then, we can pass to the limit using Lemma 27 and the fact that $\Delta(|x|^2\varphi_R(x))$ remains bounded and $\nabla(|x|^2\varphi_R(x))$ is Lipschitz continuous. ■

Chapter 4

Mathematical analysis of the origin of CTCs clusters

Abstract

Cancer cells that break away from the primary tumor and enter the bloodstream are called as circulating tumor cells (CTCs). These CTCs are suspected to be the starting point for distant metastases. Using a mathematical model, we propose to investigate whether CTCs aggregate into clusters upon leaving the primary tumor. In this chapter, we develop a Keller-Segel-type model that incorporates rotational chemotactic motion. One of the main challenges in studying this model is the lack of symmetry due to tensorial chemotaxis. However, we identify optimal conditions for radial initial data, enabling us to determine whether the solutions of the model are global or result in blow-up within a finite time. Additionally, we explore the possibility of tumor aggregation under conditions of low CTCs density. The research discussed in this chapter has been submitted for publication and is currently under review at the time of this thesis submission.

Macrophages are a type of white blood cells that are involved in the detection and destruction of harmful organisms. However, it has been proven that the interaction between macrophages and tumor cells during cell migration may contribute to the multiplication of a primary tumor to the surroundings, a process known as metastasis [62]. The communication between tumor cells and macrophages is governed by chemotactic signals. A chemical called colony-stimulating factor 1 (CSF-1) is first secreted by tumor cells. CSF-1 then binds to receptors on macrophages, causing the macrophages to produce epidermal growth factor (EGF). Subsequently, EGF binds to tumor cell receptors and further activates them. In turn, activated tumor cells release more CSF-1 and partially direct their movement to the EGF and CSF-1 concentration gradient. Detailed information on the chemotactic signaling loop between the two cell types can be found in [91]. As a result, all tumor cells aggregate and enter the bloodstream, a process called intravasation. Next, when tumor cells find a niche, the reproduction starts, and then secondary tumors could appear in distant places producing metastasis. The technical name for this mechanism is known in the scientific literature as the EGF/CSF-1 paracrine invasion loop. When additionally, the tumor cells produce an extra chemical to attract another tumor cells of the same type, it said that we have a EGF/CSF-1 autocrine invasion loop. It has been suggested that the understanding of all this dynamics could be fundamental to control metastasis with therapeutic methods [69].

Origin of CTC clusters

Cancer cells that escape from the tumor mass and pass into the bloodstream/lymphatic system are called circulating tumor cells or CTCs. It has been hypothesized that distant metastases begin with CTCs. Experiments in mice suggest that CTC clusters cannot result from intravascular aggregation (e.g., [1]). Actually, there are many conditions in the bloodstream that are hostile to epithelial cells, including shear stress, oxidative stress, and immune attacks. It has also been noted in [80] that CTCs in the bloodstream have a very short lifespan so they do not have time to accumulate in most cases. However, there is still controversy about the origin of CTC clusters. For instance, it has recently reported in [61] that aggregation of tumor cells by intravital microscopic imaging did not result from collective migration or cohesive detachment. Based on all these studies, we proposed to test through a mathematical model the possibility of CTCs forming clusters after they exit the primary tumor.

As a starting point to construct our model, we recall that in the absence of any flow, the dynamics between tumor cells and macrophages in breast cancer, was described in [54] by the system

$$\begin{aligned} \partial_t u_1 &= \mu_1 \triangle u_1 - \chi_{11} \nabla \left(u_1 \nabla v_1 \right) - \chi_{12} \nabla \left(u_1 \nabla v_2 \right) \\ \partial_t u_2 &= \mu_2 \triangle u_2 - \chi_{21} \nabla \left(u_2 \nabla v_1 \right) \\ \varepsilon_1 v_{1t} &= \triangle v_1 - v_1 + a_1 u_1, \\ \varepsilon_2 v_{2t} &= \triangle v_2 - v_2 + a_2 u_2, \end{aligned} \tag{4.1}$$

The parameters $\mu_1, \mu_2, \chi_{11}, \chi_{12}, \chi_{21}, \chi_{22}, a_1$ and a_2 represent positive constants. The variables u_1 and u_2 denote the density variables for the tumors cells and the macrofages respectively. The variable v_1 represents the concentration of chemoattractant CSF-1 and the variable v_2 denotes de concentration of the chemoattractant EGF. This mathematical model indicates that the motion of both species is driven by self-diffusion and the chemotactic response. This system can be simplified assuming that $\varepsilon_1 \approx 0$ as well as $\varepsilon_2 \approx 0$, meaning that the chemical molecular diffusion occurs at a faster rate than the diffusion of cell density [34, 50]. This simplification, led the authors of [37] to study the system

$$\partial_t u_1 = \mu_1 \triangle u_1 - \chi_{11} \nabla \left(u_1 \nabla v_1 \right) - \chi_{12} \nabla \left(u_1 \nabla v_2 \right), \tag{4.2}$$
$$\partial_t u_2 = \mu_2 \triangle u_2 - \chi_{21} \nabla \left(u_2 \nabla v_1 \right) - \chi_{22} \nabla \left(u_1 \nabla v_2 \right), \tag{4.3}$$

where

$$v_1 = -\frac{1}{2\pi} \ln |\cdot| * (a_{11} u_1 + a_{12} u_2) \text{ and } v_2 = -\frac{1}{2\pi} \ln |\cdot| * (a_{21} u_1 + a_{22} u_2). \tag{4.4}$$

It was proved that in the two-dimensional case the solutions u_1 and u_2 of system (4.2)-(4.4) can blowup in finite time. It was also found optimal conditions for having global solutions.

In comparison with the hypothesis underlying model (4.1), we want to describe the dynamics of paracrine and autocrine signalling loops when cells are surrounded by fluid. In this case, we recall that motion across thin layers should normally involve rotational components in the cross-diffusive flux ([92, 93]). Thus, we propose to introduce a rotation matrix

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \tag{4.5}$$

where $\alpha \in (-\pi, \pi]$ represents a constant and to consider the mathematical model

$$\begin{aligned} \partial_t u_1 &= \mu_1 \triangle u_1 - \chi_{11} \nabla \left(u_1 A \nabla v_1 \right) - \chi_{12} \nabla \left(u_1 A \nabla v_2 \right), \\ \partial_t u_2 &= \mu_2 \triangle u_2 - \chi_{21} \nabla \left(u_2 A \nabla v_1 \right), \end{aligned} \tag{4.6}$$

where

$$v_1 = -\frac{1}{2\pi} \ln|\cdot| * (a_{11} u_1) \text{ and } v_2 = -\frac{1}{2\pi} \ln|\cdot| * (a_{22} u_2). \tag{4.7}$$

Our results are presented here, however, through a more general system of PDEs. Namely, we consider rotation matrices

$$A_{ij} = \begin{pmatrix} \cos\alpha_{ij} & -\sin\alpha_{ij} \\ \sin\alpha_{ij} & \cos\alpha_{ij} \end{pmatrix}, \text{ for } i, j = 1, 2, \tag{4.8}$$

where the parameters $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \in (-\pi, \pi]$ are assumed to be constants and next, we propose to study the mathematical model

$$\begin{aligned} \partial_t u_1 &= \mu_1 \triangle u_1 - \chi_{11} \nabla \left(u_1 A_{11} \nabla v_1 \right) - \chi_{12} \nabla \left(u_1 A_{12} \nabla v_2 \right), \\ \partial_t u_2 &= \mu_2 \triangle u_2 - \chi_{21} \nabla \left(u_2 A_{21} \nabla v_1 \right) - \chi_{22} \nabla \left(u_2 A_{22} \nabla v_2 \right), \end{aligned} \tag{4.9}$$

where

$$v_1 = -\frac{1}{2\pi} \ln|\cdot| * (a_{11} u_1 + a_{12} u_2) \text{ and } v_2 = -\frac{1}{2\pi} \ln|\cdot| * (a_{21} u_1 + a_{22} u_2). \tag{4.10}$$

A number of technical challenges arise in analyzing this system, which we can overcome by assuming appropriate conditions for the parameters (e.g. (4.13)).

For interactions between tumor cells and macrophages to lead to group migration, cells must have a tendency to cluster. In this work, we investigate how features of the paracrine and autocrine signalling loops contribute to cell aggregation when it occurs in a rotating fluid. Essentially, we address the following questions:

- What are suitable conditions to induce cell aggregation due to chemotactic attraction? In this way, we expect to gain some understanding of the conditions under which CTCs can metastasize.
- Complementary to the previous question, we ask whether it is possible to characterize the relationship between the parameters of the model (4.8)-(4.10) to guarantee that there is no cell aggregation. We note that the answer to this question may provide insight into the development of drugs that prevent aggregation of CTCs.

- Is it possible that CTCs aggregate even if they are present only in small numbers?
- Can fluid rotation delay or prevent aggregation of CTCs?

To address these questions, we will concentrate on the two-dimensional case. We notice that the local pre-blow up behavior corresponds to biologically reasonable cell accumulation due to the chemotactic attraction, thus given a reasonable insight about the conditions allowing cells to produce metastasis. This remark motivates us to find out whether the solutions of the system (4.8)-(4.10), can blow-up or not. If this is the case, we would also like to find out what role fluid rotation plays in this phenomenon.

From a mathematical point of view, a main feature in the analysis of system (4.8)-(4.10) is that traditional approaches to constructing energy functionals (e.g [15, 36, 28, 64]) become challenging. The predominant challenge lies once more in the absence of symmetry induced by tensorial chemoattraction, complicating the treatment of entropy functionals such as $\int_{\mathbb{R}^2} u_i \log u_i dx$ with $i = 1, 2$. We will show how our approach, developed in the previous chapter, allows us to address this challenge. Specifically, we will modify these entropy functionals by introducing alternatives with lower bounds. This modification enables finding of optimal conditions on the initial data necessary for achieving global solutions. Similarly, due to tensorial chemotaxis, it is difficult to find conditions that guarantee a blow-up of the solutions. Nevertheless, we find conditions on radial initial data that allow us to decide whether the solutions of model (4.8)-(4.10) blow up within finite time. Finally, a discussion is provided on possible biological interpretation of the results.

Introducing our results, let us begin by defining a weak solution for system (4.8)-(4.10).

Definition 28 (Weak solution) *Let $\mu_1, \mu_2, \chi_{11}, \chi_{12}, \chi_{21}, \chi_{22}$ be non-negative constants, meanwhile $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$ are constants restricted to the interval $(-\pi, \pi]$ and $a_{11}, a_{12}, a_{21}, a_{22}$ arbitrary constants. Let A_{ij} with $i, j = 1, 2$, be the 2×2 matrices defined by (4.8). Given $T > 0$, the vector-valued function (u_1, u_2) is a **weak solution** on $\mathbb{R}^2 \times (0, T)$ of system (4.8)-(4.10), with initial data satisfying*

$$0 \leq u_{i0} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2),\ u_{i0} \ln u_{i0} \in L^1(\mathbb{R}^2),\ u_{i0} \ln(1 + |x|^2) \in L^1(\mathbb{R}^2), \tag{4.11}$$

for $i = 1, 2$, if

i) $u_1, u_2 \in C([0, T); L^1(\mathbb{R}^2)) \cap L^{4/3}((0, T) \times L^{4/3}(\mathbb{R}^2))$, and

ii) (u_1, u_2) verify (4.9) in the weak sense, that is to say

$$\begin{aligned} &\int_{\mathbb{R}^2} \varphi u_i(x, t) dx + \int_{\mathbb{R}^2} \varphi u_{i0}(x) dx \\ &= \mu_i \int_0^t \int_{\mathbb{R}^2} u_i \Delta\varphi dx d\tau + \chi_{i1} \int_0^t \int_{\mathbb{R}^2} \nabla\varphi \cdot (u_i A_{i1} \nabla v_1)\, dx d\tau \\ &+ \chi_{i2} \int_0^t \int_{\mathbb{R}^2} \nabla\varphi \cdot (u_i A_{i2} \nabla v_2)\, dx d\tau, \end{aligned}$$

for $i = 1, 2$, for any $\varphi \in C_0^\infty(\mathbb{R}^2)$, $0 < t < T$, and for v_1, v_2 being defined by (4.10).

The main result of this chapter is as follows:

Theorem 29 ***(Sharp conditions, finite time blow-up and global existence)*** *Let u_{10}, u_{20} satisfying (4.11), then there exists a maximal time $T_{\max} > 0$ of existence of a positive weak solution (u_1, u_2) to the system (4.8)-(4.10). Let us denote by θ_i with $i = 1, 2$ the total initial masses define by*

$$\theta_1 := \int_{\mathbb{R}^2} u_{10} dx > 0 \quad and \quad \theta_2 := \int_{\mathbb{R}^2} u_{20} dx > 0. \tag{4.12}$$

Assuming that $\chi_{ij}, a_{ij}, \alpha_{ij}, i = 1, 2$, satisfy

$$\begin{aligned}
&\delta_{11} := a_{11}\chi_{11}\cos\alpha_{11} + a_{21}\chi_{12}\cos\alpha_{12} \geq 0,\\
&\delta_{22} := a_{12}\chi_{21}\cos\alpha_{21} + a_{22}\chi_{22}\cos\alpha_{22} \geq 0,\\
&\delta_{12} := a_{12}\chi_{11}\cos\alpha_{11} + a_{22}\chi_{12}\cos\alpha_{12} > 0,\\
&\delta_{21} := a_{11}\chi_{21}\cos\alpha_{21} + a_{21}\chi_{22}\cos\alpha_{22} > 0,\\
&a_{11}a_{22} - a_{12}a_{21} \neq 0,\ \ a_{11}a_{12} \geq 0, a_{21}a_{22} \geq 0\\
&and\ \chi_{11}\chi_{12}\sin(\alpha_{12} - \alpha_{11}) = \chi_{21}\chi_{22}\sin(\alpha_{22} - \alpha_{21}) = 0,
\end{aligned} \tag{4.13}$$

we have the following conclusions:

1. *If (θ_1, θ_2) satisfies*

$$\begin{aligned}
&\delta_{11}\theta_1 < 8\pi\mu_1,\ \ \delta_{22}\theta_2 < 8\pi\mu_2,\\
and\ &\frac{8\pi\mu_1}{\delta_{12}}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right) > 0.
\end{aligned} \tag{4.14}$$

then, $T_{\max} = +\infty$. Moreover, assumming $u_{10}, u_{20} \in L^1(R^2, (1 + |x|^2)dx)$, we have that the free-energy functional defined by

$$\begin{aligned}
&E(t) \\
&:= \frac{1}{\delta_{12}}\left(\int_{\mathbb{R}^2} \mu_1 u_1 \ln u_1 dx - \frac{1}{2}\int_{\mathbb{R}^2} u_1(\chi_{11}\cos\alpha_{11}v_1 + \chi_{12}\cos\alpha_{12}v_2)dx\right)\\
&+ \frac{1}{\delta_{21}}\left(\int_{\mathbb{R}^2} \mu_2 u_2 \ln u_2 dx - \frac{1}{2}\int_{\mathbb{R}^2} u_2(\chi_{21}\cos\alpha_{21}v_1 + \chi_{22}\cos\alpha_{22}v_2)dx\right).
\end{aligned} \tag{4.15}$$

satisfies the following dissipation inequality

$$\begin{aligned}
&E(t)\\
&+ \frac{1}{\delta_{12}}\int_{\mathbb{R}^2} u_1^\varepsilon \left|\nabla(\mu_1 \ln u_1^\varepsilon - \chi_{11}\cos\alpha_{11}v_1^\varepsilon - \chi_{12}\cos\alpha_{12}v_2^\varepsilon))\right|^2 dx\\
&+ \frac{1}{\delta_{21}}\int_{\mathbb{R}^2} u_2^\varepsilon \left|\nabla(\mu_2 \ln u_2^\varepsilon - \chi_{21}\cos\alpha_{21}v_1^\varepsilon - \chi_{22}\cos\alpha_{22}v_2^\varepsilon))\right|^2 dx\\
&\leq E(0).
\end{aligned} \tag{4.16}$$

2. *Assume that the initial data* u_{10}, u_{20} *are radially symmetric and* $u_{10}, u_{20} \in L^1(\mathbb{R}^2, (1+|x|^2)dx)$. *If* θ_1 *and* θ_2 *satisfy any of the inequalities*

$$\begin{aligned} &\delta_{11}\theta_1 > 8\pi\mu_1, \ \textit{or} \ \delta_{22}\theta_2 > 8\pi\mu_2, \\ \textit{or} \ &\frac{8\pi\mu_1}{\delta_{12}}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right) < 0 \end{aligned} \tag{4.17}$$

then $T_{\max} < +\infty$.
On the other hand, assuming that $\chi_{ij}, a_{ij}, \alpha_{ij}, i = 1, 2,$ *satisfy*

$$\delta_{11} \le 0, \delta_{22} \le 0, \delta_{12} \le 0, \delta_{21} \le 0, a_{11}a_{12} \ge 0, a_{21}a_{22} \ge 0, \tag{4.18}$$

we have that for any initial masses θ_i, $i = 1, 2$, *it holds* $T_{\max} = +\infty$.

We note that although restrictions (4.13) and (4.18) on the parameters seem to be very restrictive at first sight, it still include several important cases. For instance the case where rotation angles satisfy $\alpha_{12} = \alpha_{11} = \alpha_{22} = \alpha_{21}$, and the coefficients $a_{12} = a_{21} = 0$, corresponds to the model (4.5)-(4.7) describing the dynamics between CTCs and macrophages.

Another interesting case is when $\chi_{12} = \chi_{21} = a_{11} = a_{22} = 0$. Then we obtain the following model describing the interaction of two cell populations undergoing rotation

$$\begin{aligned} \partial_t u_1 &= \mu_1 \triangle u_1 - \chi_{11}\nabla \cdot (u_1 A_{11} \nabla v_1), \\ \partial_t u_2 &= \mu_2 \triangle u_2 - \chi_{22}\nabla \cdot (u_2 A_{22} \nabla v_2), \\ v_1 &= -\tfrac{1}{2\pi} \ln|\cdot| * (a_{12}u_2), \\ v_2 &= -\tfrac{1}{2\pi} \ln|\cdot| * (a_{21}u_1). \end{aligned} \tag{4.19}$$

On the other hand, if $\sin(\alpha_{i2} - \alpha_{i1}) = 0, i = 1, 2$, or equivalently $\alpha_{i2} = k\pi + \alpha_{i1}, k \in \mathbb{Z}, i = 1, 2$, we have that there are two possible scenarios: Either the cell's population u_i has a dynamic of cooperating effects of attraction (or repulsion) in chemotaxis or competing effects of attraction vs. repulsion in chemotaxis.

Remark 30 *Assuming that* $\chi_{ij}, a_{ij}, \alpha_{ij}, i = 1, 2,$ *satisfy (4.13). Let us consider the conic section*

$$\frac{8\pi\mu_1}{\delta_{12}}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right) = 0.$$

The discriminant D *is given by*

$$D = 4\left(1 - \frac{\delta_{11}\delta_{22}}{\delta_{12}\delta_{21}}\right).$$

Notice that

$$\begin{aligned} &\delta_{11}\delta_{22} - \delta_{12}\delta_{21} \\ &= \begin{vmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{vmatrix} \\ &= \begin{vmatrix} \chi_{11}\cos\alpha_{11} & \chi_{12}\cos\alpha_{12} \\ \chi_{21}\cos\alpha_{21} & \chi_{22}\cos\alpha_{22} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= (\chi_{11}\chi_{22}\cos\alpha_{11}\cos\alpha_{22} - \chi_{12}\chi_{21}\cos\alpha_{12}\cos\alpha_{21})\underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{\neq 0}. \end{aligned}$$

It follows that, we will have a parabola when

$$\chi_{11}\chi_{22}\cos\alpha_{11}\cos\alpha_{22} = \chi_{12}\chi_{21}\cos\alpha_{12}\cos\alpha_{21},$$

we will have a ellipse when

$$\begin{gathered} a_{11}a_{22} - a_{12}a_{21} > 0 \\ \text{and } \chi_{11}\chi_{22}\cos\alpha_{11}\cos\alpha_{22} > \chi_{12}\chi_{21}\cos\alpha_{12}\cos\alpha_{21}, \end{gathered}$$

or

$$\begin{gathered} a_{11}a_{22} - a_{12}a_{21} < 0 \\ \text{and } \chi_{11}\chi_{22}\cos\alpha_{11}\cos\alpha_{22} < \chi_{12}\chi_{21}\cos\alpha_{12}\cos\alpha_{21}, \end{gathered}$$

and we will have a hyperbola when

$$\begin{gathered} a_{11}a_{22} - a_{12}a_{21} > 0 \\ \text{and } \chi_{11}\chi_{22}\cos\alpha_{11}\cos\alpha_{22} < \chi_{12}\chi_{21}\cos\alpha_{12}\cos\alpha_{21}, \end{gathered}$$

or

$$\begin{gathered} a_{11}a_{22} - a_{12}a_{21} < 0 \\ \text{and } \chi_{11}\chi_{22}\cos\alpha_{11}\cos\alpha_{22} > \chi_{12}\chi_{21}\cos\alpha_{12}\cos\alpha_{21}. \end{gathered}$$

Remark 31 *Note that if $\alpha_{ij} = 0, i, j = 1, 2$ and $\mu_1 = \mu_2 = 1$, we rescue the two-species Keller-Segel model with two chemicals proposed in [37]*

$$\begin{array}{ll} \partial_t u_1 = \triangle u_1 - \chi_{11}\nabla\cdot(u_1\nabla v_1) - \chi_{12}\nabla\cdot(u_1\nabla v_2), & x\in\mathbb{R}^2, t>0, \\ \partial_t u_2 = \triangle u_2 - \chi_{21}\nabla\cdot(u_2\nabla v_1) - \chi_{22}\nabla\cdot(u_2\nabla v_2), & x\in\mathbb{R}^2, t>0, \\ -\triangle v_1 = a_{11}u_1 + a_{12}u_2, & x\in\mathbb{R}^2, t>0, \\ -\triangle v_2 = a_{21}u_1 + a_{22}u_2, & x\in\mathbb{R}^2, t>0. \end{array}$$

Theorem 29 guarantees that if the initial masses θ_1 and θ_2 satisfy

$$\begin{gathered} \frac{8\pi}{\chi_{11}a_{12}+\chi_{12}a_{22}}\theta_1 + \frac{8\pi}{\chi_{21}a_{11}+\chi_{22}a_{21}}\theta_2 \\ -\left(\frac{\chi_{11}a_{11}+\chi_{12}a_{21}}{\chi_{11}a_{12}+\chi_{12}a_{22}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\chi_{21}a_{12}+\chi_{22}a_{22}}{\chi_{21}a_{11}+\chi_{22}a_{21}}\theta_2^2\right) > 0, \\ \theta_1 < \frac{8\pi}{\chi_{11}a_{11}+\chi_{12}a_{21}} \text{ and } \theta_2 < \frac{8\pi}{\chi_{21}a_{12}+\chi_{22}a_{22}}, \end{gathered}$$

then the corresponding solution exists globally in time. Moreover, we can always construct initial data with masses θ_1 and θ_2 such that if they satisfy any of the inequalities

$$\begin{gathered} \frac{8\pi}{\chi_{11}a_{12}+\chi_{12}a_{22}}\theta_1 + \frac{8\pi}{\chi_{21}a_{11}+\chi_{22}a_{21}}\theta_2 \\ -\left(\frac{\chi_{11}a_{11}+\chi_{12}a_{21}}{\chi_{11}a_{12}+\chi_{12}a_{22}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\chi_{21}a_{12}+\chi_{22}a_{22}}{\chi_{21}a_{11}+\chi_{22}a_{21}}\theta_2^2\right) < 0, \\ \text{or } \theta_1 > \frac{8\pi}{\chi_{11}a_{11}+\chi_{12}a_{21}}, \text{ or } \theta_2 > \frac{8\pi}{\chi_{21}a_{12}+\chi_{22}a_{22}}, \end{gathered}$$

then $T_{\max} < \infty$. These results coincide with the sharp result given in [37].

Remark 32 *Note that if* $\alpha_{ij} = 0, i, j = 1, 2$, $\mu_1 = \mu_2 = 1$, $\chi_{11} = \chi_{22} = 1, \chi_{12} = \chi_{21} = 0, a_{11} = a_{22} = 0, a_{12} = a_{21} = 1$,*we obtain as a particular case the two-species Keller-Segel model with two chemicals, that describe the competition of two species, discussed in [49]*

$$\begin{aligned} &\partial_t u_1 = \triangle u_1 - \nabla \cdot (u_1 \nabla v_1), && x \in \mathbb{R}^2, t > 0, \\ &\partial_t u_2 = \triangle u_2 - \nabla \cdot (u_2 \nabla v_2), && x \in \mathbb{R}^2, t > 0, \\ &-\triangle v_1 = u_2, && x \in \mathbb{R}^2, t > 0, \\ &-\triangle v_2 = u_1, && x \in \mathbb{R}^2, t > 0. \end{aligned}$$

Theorem 29 guarantees that if the initial masses θ_1 *and* θ_2 *satisfy*

$$\theta_1\theta_2 - 4\pi(\theta_1 + \theta_2) < 0,$$

then the corresponding solution exists globally in time. Moreover, we can always construct initial data with masses θ_1 *and* θ_2 *such that if they satisfy the inequality*

$$\theta_1\theta_2 - 4\pi(\theta_1 + \theta_2) > 0,$$

then $T_{\max} < \infty$. *These results coincide with the sharp result given in [49].*

4.1 Global existence

Our purpose in this section is to prove the following global existence results

Theorem 33 *Assume that* u_{10}, u_{20} *satisfy (4.11) and* $\chi_{ij}, a_{ij}, \alpha_{ij}, i = 1, 2$, *satisfy (4.13). Let us denote by* θ_i *with* $i = 1, 2$ *the total initial masses define by (4.12). If* (θ_1, θ_2) *satisfies*

$$\begin{aligned} &\delta_{11}\theta_1 < 8\pi\mu_1, \ \delta_{22}\theta_2 < 8\pi\mu_2, \\ \text{and } &\frac{8\pi\mu_1}{\delta_{12}}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right) > 0, \end{aligned} \tag{4.20}$$

then system (4.8)-(4.10) has a global weak solution satisfying the energy dissipation (4.16) under the additional hypothesis $u_{10}|x|^2, u_{20}|x|^2 \in L^1(\mathbb{R}^2)$.

Theorem 34 *Assume that* u_{10}, u_{20} *satisfy (4.11) and* $\chi_{ij}, a_{ij}, \alpha_{ij}, i = 1, 2$, *satisfy (4.18), then for any initial masses* θ_i, $i = 1, 2$, *the system (4.8)-(4.10) has a global weak solution.*

Our approach to proving Theorem 33 relies heavily on the technique introduced in the Chapter 3, which is designed for analyzing the global existence of the multi-species Keller-Segel model with rotational flux. This approach can be outlined in three parts: firstly, we constructed a regularized version of the system (4.8)-(4.10) having smooth solutions and introduced some of its properties like mass conservation, integrability, and positivity. Secondly, we showed how to obtain uniform estimates of the regularized system to pass to the limit, and obtained the result of global existence of weak solutions for the system (4.8)-(4.10) and, finally, we showed that the weak solutions of the system (4.8)-(4.10) satisfy the free-energy inequality (4.16). On the other hand, in the proof of the theorem 34, it is not necessary the use of any energy functional and instead, a direct approach is enough to bound the L^p-norms in time.

4.1.1 Proof of theorem 33

Regularized Problem and some important properties We consider the regularized problem for $0 < \varepsilon < 1/\sqrt{2}$

$$\begin{array}{ll} \partial_t u_1^\varepsilon = \mu_1 \triangle u_1^\varepsilon - \chi_{11}\nabla\cdot(u_1^\varepsilon A_{11}\nabla v_1^\varepsilon) - \chi_{12}\nabla\cdot(u_1^\varepsilon A_{12}\nabla v_2^\varepsilon), & x\in\mathbb{R}^2, t>0,\\ \partial_t u_2^\varepsilon = \mu_2 \triangle u_2^\varepsilon - \chi_{21}\nabla\cdot(u_2^\varepsilon A_{21}\nabla v_1^\varepsilon) - \chi_{22}\nabla\cdot(u_2^\varepsilon A_{22}\nabla v_2^\varepsilon), & x\in\mathbb{R}^2, t>0,\\ v_1^\varepsilon = \mathbf{K}^\varepsilon * (a_{11}u_1^\varepsilon + a_{12}u_2^\varepsilon), & x\in\mathbb{R}^2, t>0,\\ v_2^\varepsilon = \mathbf{K}^\varepsilon * (a_{21}u_1^\varepsilon + a_{22}u_2^\varepsilon), & x\in\mathbb{R}^2, t>0,\\ u_1^\varepsilon(x,0) = u_{10}^\varepsilon(x), u_2^\varepsilon(x,0) = u_{20}^\varepsilon(x) \in L^1_+(\mathbb{R}^2) & x\in\mathbb{R}^2. \end{array} \tag{4.21}$$

Here $\mathbf{K}^\varepsilon$ is defined as 3.50. By applying the same argument as in Proposition 14, we extend the result concerning the global existence of smooth solutions and some important properties to our scenario involving two types of chemicals, rather than just one. Therefore, we omit the proof here.

Proposition 35 *Assume that u_{10}, u_{20} satisfy (4.11) and (4.12). Then there is a unique classic solution $(u_1^\varepsilon, u_2^\varepsilon) \in BC([0,T); L^1(\mathbb{R}^2)) \cap C^{2,1}(\mathbb{R}^2\times(0,T)) \cap X_T$, of (4.21) with $0<\varepsilon<1/\sqrt{2}$ on $[0,T)$, for any $0<T<\infty$. Then $(u_1^\varepsilon, u_2^\varepsilon)$ satisfies the following properties:*

(i) *mass conservation, i.e., $\int_{\mathbb{R}^2} u_i^\varepsilon(x,t)dx = \theta_i$, for $i=1,2$ and $t\in[0,T)$;*

(ii) *integrability, i.e., for every $1\le p\le\infty$, there holds $u_i^\varepsilon \in L^\infty((0,T); L^p(\mathbb{R}^2))$ for $i=1,2$;*

(iii) *positivity, i.e., $u_i^\varepsilon > 0$ for all $(x,t)\in\mathbb{R}^2\times(0,T)$ for $i=1,2$;*

(iv) *$u_i^\varepsilon \ln(1+|x|^2) \in L^\infty((0,T); L^1(\mathbb{R}^2))$ for $i=1,2$;*

(v) *$u_i^\varepsilon \ln u_i^\varepsilon \in L^\infty((0,T); L^1(\mathbb{R}^2))$ for $i=1,2$;*

(vi) *$|\nabla(\sqrt{u_i^\varepsilon})| \in L^2((0,T); L^2(\mathbb{R}^2))$ for $i=1,2$;*

(vii) *$u_i^\varepsilon v_j^\varepsilon \in L^\infty((0,T); L^1(\mathbb{R}^2))$ for $i,j=1,2$;*

(viii) *for every $2<p<\infty$, there holds $\nabla v_i^\varepsilon \in L^\infty((0,T); W^{1,p}(\mathbb{R}^2))^2$* for $i=1,2$.

Dissipative energy structure. A main tool to analyze the qualitative behavior of the solutions to the models, we are working with, are the free-energy functionals. The derivation of such kinds of functionals is especially challenging in the multispecies case due to the lack of symmetry arising from the different angles of rotation. We are now going to show the existence of free-energy functionals when the parameters $\chi_{ij}, a_{ij}, \alpha_{ij}, i=1,2$, satisfy (4.13).

We define the free-energy functional $E_\varepsilon(t)$ associated to system (4.21) as

$$\begin{aligned} &E_\varepsilon(t)\\ &:= \frac{1}{\delta_{12}}\left(\int_{\mathbb{R}^2}\mu_1 u_1^\varepsilon \ln u_1^\varepsilon dx - \frac{1}{2}\int_{\mathbb{R}^2} u_1^\varepsilon(\chi_{11}\cos\alpha_{11}v_1^\varepsilon + \chi_{12}\cos\alpha_{12}v_2^\varepsilon)dx\right)\\ &+ \frac{1}{\delta_{21}}\left(\int_{\mathbb{R}^2}\mu_2 u_2^\varepsilon \ln u_2^\varepsilon dx - \frac{1}{2}\int_{\mathbb{R}^2} u_2^\varepsilon(\chi_{21}\cos\alpha_{21}v_1^\varepsilon + \chi_{22}\cos\alpha_{22}v_2^\varepsilon)dx\right). \end{aligned} \tag{4.22}$$

Now, we show that the free energy functional $E_\varepsilon(t)$ enjoys a basic energy law such that it is monotone non-increasing with respect to time.

Theorem 36 *Let $(u_1^\varepsilon, u_2^\varepsilon)$ be a classical solution of system (4.21). Then,*

$$\begin{aligned}
&\frac{d}{dt}E_\varepsilon(t)\\
&= -\frac{1}{\delta_{12}}\int_{\mathbb{R}^2} u_1^\varepsilon \left|\nabla(\mu_1 \ln u_1^\varepsilon - \chi_{11}\cos\alpha_{11}v_1^\varepsilon - \chi_{12}\cos\alpha_{12}v_2^\varepsilon)\right|^2 dx\\
&- \frac{1}{\delta_{21}}\int_{\mathbb{R}^2} u_2^\varepsilon \left|\nabla(\mu_2 \ln u_2^\varepsilon - \chi_{21}\cos\alpha_{21}v_1^\varepsilon - \chi_{22}\cos\alpha_{22}v_2^\varepsilon)\right|^2 dx. \qquad (4.23)
\end{aligned}$$

for all $t > 0$.

Proof. *By hypothesis we have that $a_{11}a_{22} - a_{12}a_{21} \neq 0$, thus, we can re-write the equations for v_1^ε and v_2^ε of (4.21) in the form*

$$-\triangle\left[\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}^{-1}\begin{pmatrix} v_1^\varepsilon\\ v_2^\varepsilon\end{pmatrix}\right] = -\triangle\mathbf{K}^\varepsilon * \begin{pmatrix} u_1^\varepsilon\\ u_2^\varepsilon\end{pmatrix}.$$

Note that

$$\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}\begin{pmatrix} a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{pmatrix}.$$

Let us define

$$\begin{pmatrix} w_1^\varepsilon\\ w_2^\varepsilon\end{pmatrix} := \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}^{-1}\begin{pmatrix} v_1^\varepsilon\\ v_2^\varepsilon\end{pmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}\begin{pmatrix} a_{22}v_1^\varepsilon - a_{12}v_2^\varepsilon\\ -a_{21}v_1^\varepsilon + a_{11}v_2^\varepsilon\end{pmatrix}.$$

Moreover,

$$\begin{pmatrix} v_1^\varepsilon\\ v_2^\varepsilon\end{pmatrix} = \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} w_1^\varepsilon\\ w_2^\varepsilon\end{pmatrix} = \begin{pmatrix} a_{11}w_1^\varepsilon + a_{12}w_2^\varepsilon\\ a_{21}w_1^\varepsilon + a_{22}w_2^\varepsilon\end{pmatrix}.$$

Then system (4.21) takes the form

$$\begin{aligned}
\partial_t u_1^\varepsilon &= \mu_1\triangle u_1^\varepsilon - \chi_{11}\nabla\cdot(u_1^\varepsilon A_{11}\nabla(a_{11}w_1^\varepsilon + a_{12}w_2^\varepsilon)) && \\
&\quad -\chi_{12}\nabla\cdot(u_1^\varepsilon A_{12}\nabla(a_{21}w_1^\varepsilon + a_{22}w_2^\varepsilon)), && x\in\mathbb{R}^2, t>0,\\
\partial_t u_2^\varepsilon &= \mu_2\triangle u_2^\varepsilon - \chi_{21}\nabla\cdot(u_2^\varepsilon A_{21}\nabla(a_{11}w_1^\varepsilon + a_{12}w_2^\varepsilon)) && \\
&\quad -\chi_{22}\nabla\cdot(u_2^\varepsilon A_{22}\nabla(a_{21}w_1^\varepsilon + a_{22}w_2^\varepsilon)), && x\in\mathbb{R}^2, t>0, \qquad (4.24)\\
-\triangle w_1^\varepsilon &= -\triangle\mathbf{K}^\varepsilon * u_1^\varepsilon, && x\in\mathbb{R}^2, t>0,\\
-\triangle w_2^\varepsilon &= -\triangle\mathbf{K}^\varepsilon * u_2^\varepsilon, && x\in\mathbb{R}^2, t>0.
\end{aligned}$$

On the other hand, we decompose the matrices $A_{ij}, i,j = 1,2$, in the form

$$A_{ij} = \begin{pmatrix} \cos\alpha_{ij} & -\sin\alpha_{ij}\\ \sin\alpha_{ij} & \cos\alpha_{ij}\end{pmatrix} = \cos\alpha_{ij}I + \sin\alpha_{ij}R, \qquad (4.25)$$

where I denotes the identity matrix and

$$R := \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}.$$

Using the first equation in (4.24) and the decomposition (4.25), we obtain

$$
\begin{aligned}
\partial_t u_1^\varepsilon &= \mu_1 \triangle u_1^\varepsilon - a_{11}\chi_{11}\nabla\cdot(u_1^\varepsilon A_{11}\nabla w_1^\varepsilon) - a_{12}\chi_{11}\nabla\cdot(u_1^\varepsilon A_{11}\nabla w_2^\varepsilon)\\
&- a_{21}\chi_{12}\nabla\cdot(u_1^\varepsilon A_{12}\nabla w_1^\varepsilon) - a_{22}\chi_{12}\nabla\cdot(u_1^\varepsilon A_{12}\nabla w_2^\varepsilon)\\
&= \mu_1\triangle u_1^\varepsilon - (a_{11}\chi_{11}\cos\alpha_{11} + a_{21}\chi_{12}\cos\alpha_{12})\nabla\cdot(u_1^\varepsilon\nabla w_1^\varepsilon)\\
&- (a_{12}\chi_{11}\cos\alpha_{11} + a_{22}\chi_{12}\cos\alpha_{12})\nabla\cdot(u_1^\varepsilon\nabla w_2^\varepsilon)\\
&- (a_{11}\chi_{11}\sin\alpha_{11} + a_{21}\chi_{12}\sin\alpha_{12})\nabla\cdot(u_1^\varepsilon\nabla^\perp w_1^\varepsilon)\\
&- (a_{12}\chi_{11}\sin\alpha_{11} + a_{22}\chi_{12}\sin\alpha_{12})\nabla\cdot(u_1^\varepsilon\nabla^\perp w_2^\varepsilon). \qquad (4.26)
\end{aligned}
$$

In order to simplify the notation, we define the constants

$$
\begin{pmatrix}\delta_{11} & \delta_{12}\\ \delta_{13} & \delta_{14}\end{pmatrix} := \begin{pmatrix}\chi_{11}\cos\alpha_{11} & \chi_{12}\cos\alpha_{12}\\ \chi_{11}\sin\alpha_{11} & \chi_{12}\sin\alpha_{12}\end{pmatrix}\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}, \qquad (4.27)
$$

and

$$
\begin{pmatrix}\delta_{21} & \delta_{22}\\ \delta_{23} & \delta_{24}\end{pmatrix} := \begin{pmatrix}\chi_{21}\cos\alpha_{21} & \chi_{22}\cos\alpha_{22}\\ \chi_{21}\sin\alpha_{21} & \chi_{22}\sin\alpha_{22}\end{pmatrix}\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}. \qquad (4.28)
$$

Then the equation (4.26) becomes

$$
\begin{aligned}
\partial_t u_1^\varepsilon &= \mu_1\triangle u_1^\varepsilon - \delta_{11}\nabla\cdot(u_1^\varepsilon\nabla w_1^\varepsilon) - \delta_{12}\nabla\cdot(u_1^\varepsilon\nabla w_2^\varepsilon)\\
&- \delta_{13}\nabla\cdot(u_1^\varepsilon\nabla^\perp w_1^\varepsilon) - \delta_{14}\nabla\cdot(u_1^\varepsilon\nabla^\perp w_2^\varepsilon).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\partial_t u_2^\varepsilon &= \mu_2\triangle u_2^\varepsilon - \delta_{21}\nabla\cdot(u_2^\varepsilon\nabla w_1^\varepsilon) - \delta_{22}\nabla\cdot(u_2^\varepsilon\nabla w_2^\varepsilon)\\
&- \delta_{23}\nabla\cdot(u_2^\varepsilon\nabla^\perp w_1^\varepsilon) - \delta_{24}\nabla\cdot(u_2^\varepsilon\nabla^\perp w_2^\varepsilon).
\end{aligned}
$$

Therefore the system (4.24) takes the equivalent form

$$
\begin{cases}
\partial_t u_1^\varepsilon = \mu_1\triangle u_1^\varepsilon - \delta_{11}\nabla\cdot(u_1^\varepsilon\nabla w_1^\varepsilon) - \delta_{12}\nabla\cdot(u_1^\varepsilon\nabla w_2^\varepsilon) & \\
\quad -\delta_{13}\nabla\cdot(u_1^\varepsilon\nabla^\perp w_1^\varepsilon) - \delta_{14}\nabla\cdot(u_1^\varepsilon\nabla^\perp w_2^\varepsilon), & x\in\mathbb{R}^2, t>0,\\
\partial_t u_2^\varepsilon = \mu_2\triangle u_2^\varepsilon - \delta_{21}\nabla\cdot(u_2^\varepsilon\nabla w_1^\varepsilon) - \delta_{22}\nabla\cdot(u_2^\varepsilon\nabla w_2^\varepsilon) & \\
\quad -\delta_{23}\nabla\cdot(u_2^\varepsilon\nabla^\perp w_1^\varepsilon) - \delta_{24}\nabla\cdot(u_2^\varepsilon\nabla^\perp w_2^\varepsilon), & x\in\mathbb{R}^2, t>0,\\
-\triangle w_1^\varepsilon = -\triangle\mathbf{K}^\varepsilon * u_1^\varepsilon, & x\in\mathbb{R}^2, t>0,\\
-\triangle w_2^\varepsilon = -\triangle\mathbf{K}^\varepsilon * u_2^\varepsilon, & x\in\mathbb{R}^2, t>0.
\end{cases} \qquad (4.29)
$$

Note that

$$
\begin{aligned}
\partial_t u_1^\varepsilon &= \nabla\cdot(\mu_1\nabla u_1^\varepsilon - \delta_{11}u_1^\varepsilon\nabla w_1^\varepsilon - \delta_{12}u_1^\varepsilon\nabla w_2^\varepsilon\\
&- \delta_{13}u_1^\varepsilon\nabla^\perp w_1^\varepsilon - \delta_{14}u_1^\varepsilon\nabla^\perp w_2^\varepsilon\\
&= \nabla\cdot(u_1^\varepsilon\nabla(\mu_1\ln u_1^\varepsilon - \delta_{11}w_1^\varepsilon - \delta_{12}w_2^\varepsilon)\\
&- \nabla\cdot(\delta_{13}u_1^\varepsilon\nabla^\perp w_1^\varepsilon + \delta_{14}u_1^\varepsilon\nabla^\perp w_2^\varepsilon). \qquad (4.30)
\end{aligned}
$$

Multiplying (4.30) by $\mu_1\ln u_1^\varepsilon - \delta_{11}w_1^\varepsilon - \delta_{12}w_2^\varepsilon$ *and integrating over* $\mathbb{R}^2$

$$
\begin{aligned}
&\int_{\mathbb{R}^2}(\partial_t u_1^\varepsilon)(\mu_1\ln u_1^\varepsilon - \delta_{11}w_1^\varepsilon - \delta_{12}w_2^\varepsilon)dx\\
&= \int_{\mathbb{R}^2}(\mu_1\ln u_1^\varepsilon - \delta_{11}w_1^\varepsilon - \delta_{12}w_2^\varepsilon)\nabla\cdot(u_1^\varepsilon\nabla(\mu_1\ln u_1^\varepsilon - \delta_{11}w_1^\varepsilon - \delta_{12}w_2^\varepsilon))dx\\
&- \int_{\mathbb{R}^2}(\mu_1\ln u_1^\varepsilon - \delta_{11}w_1^\varepsilon - \delta_{12}w_2^\varepsilon)\nabla\cdot(\delta_{13}u_1^\varepsilon\nabla^\perp w_1^\varepsilon + \delta_{14}u_1^\varepsilon\nabla^\perp w_2^\varepsilon)dx.
\end{aligned}
$$

Integrating by parts, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2}(\partial_t u_1^\varepsilon)(\mu_1\ln u_1^\varepsilon-\delta_{11}w_1^\varepsilon-\delta_{12}w_2^\varepsilon)dx\\
&=-\int_{\mathbb{R}^2}\nabla(\mu_1\ln u_1^\varepsilon-\delta_{11}w_1^\varepsilon-\delta_{12}w_2^\varepsilon)\cdot(u_1^\varepsilon\nabla(\mu_1\ln u_1^\varepsilon-\delta_{11}w_1^\varepsilon-\delta_{12}w_2^\varepsilon))dx\\
&+\int_{\mathbb{R}^2}\nabla(\mu_1\ln u_1^\varepsilon-\delta_{11}w_1^\varepsilon-\delta_{12}w_2^\varepsilon)\cdot(\delta_{13}u_1^\varepsilon\nabla^\perp w_1^\varepsilon+\delta_{14}u_1^\varepsilon\nabla^\perp w_2^\varepsilon)dx\\
&=-\int_{\mathbb{R}^2}u_1^\varepsilon\left|\nabla(\mu_1\ln u_1^\varepsilon-\delta_{11}w_1^\varepsilon-\delta_{12}w_2^\varepsilon)\right|^2dx\\
&-\mu_1\delta_{13}\int_{\mathbb{R}^2}u_1^\varepsilon\nabla\cdot\nabla^\perp w_1^\varepsilon-\mu_1\delta_{14}\int_{\mathbb{R}^2}u_1^\varepsilon\nabla\cdot\nabla^\perp w_2^\varepsilon\\
&-\delta_{11}\delta_{13}\int_{\mathbb{R}^2}u_1^\varepsilon\nabla w_1^\varepsilon\cdot\nabla^\perp w_1^\varepsilon-\delta_{11}\delta_{14}\int_{\mathbb{R}^2}u_1^\varepsilon\nabla w_1^\varepsilon\cdot\nabla^\perp w_2^\varepsilon\\
&-\delta_{12}\delta_{13}\int_{\mathbb{R}^2}u_1^\varepsilon\nabla w_2^\varepsilon\cdot\nabla^\perp w_1^\varepsilon\\
&-\delta_{12}\delta_{14}\int_{\mathbb{R}^2}u_1^\varepsilon\nabla w_2^\varepsilon\cdot\nabla^\perp w_2^\varepsilon dx.
\end{aligned}$$

Using $\nabla\cdot\nabla^\perp w_i^\varepsilon=\nabla w_i^\varepsilon\cdot\nabla^\perp w_i^\varepsilon=0, i=1,2$ *and* $\nabla w_2^\varepsilon\cdot\nabla^\perp w_1^\varepsilon=-\nabla w_1^\varepsilon\cdot\nabla^\perp w_2^\varepsilon$, *we have that*

$$\begin{aligned}
&\int_{\mathbb{R}^2}(\partial_t u_1^\varepsilon)(\mu_1\ln u_1^\varepsilon-\delta_{11}w_1^\varepsilon-\delta_{12}w_2^\varepsilon)dx\\
&=-\int_{\mathbb{R}^2}u_1^\varepsilon\left|\nabla(\mu_1\ln u_1^\varepsilon-\delta_{11}w_1^\varepsilon-\delta_{12}w_2^\varepsilon)\right|^2dx\\
&-(\delta_{11}\delta_{14}-\delta_{12}\delta_{13})\int_{\mathbb{R}^2}u_1^\varepsilon\nabla w_1^\varepsilon\cdot\nabla^\perp w_2^\varepsilon dx
\end{aligned}$$

At this point, we notice that the assumption $\chi_{11}\chi_{12}\sin(\alpha_{12}-\alpha_{11})=0$ *allow as to simplify the expression* $\delta_{11}\delta_{14}-\delta_{12}\delta_{13}$ *just by taking determinants in (4.27) to obtain*

$$\begin{aligned}
&\delta_{11}\delta_{14}-\delta_{12}\delta_{13}\\
&=\begin{vmatrix}\delta_{11}&\delta_{12}\\ \delta_{13}&\delta_{14}\end{vmatrix}=\begin{vmatrix}\chi_{11}\cos\alpha_{11}&\chi_{12}\cos\alpha_{12}\\ \chi_{11}\sin\alpha_{11}&\chi_{12}\sin\alpha_{12}\end{vmatrix}\begin{vmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{vmatrix}\\
&=\chi_{11}\chi_{12}(\cos\alpha_{11}\sin\alpha_{12}-\cos\alpha_{12}\sin\alpha_{11})(a_{11}a_{22}-a_{12}a_{21})\\
&=\chi_{11}\chi_{12}\sin(\alpha_{12}-\alpha_{11})(a_{11}a_{22}-a_{12}a_{21})=0 \qquad (4.31)
\end{aligned}$$

In conclusion

$$\begin{aligned}
&\int_{\mathbb{R}^2}(\partial_t u_1^\varepsilon)(\mu_1\ln u_1^\varepsilon-\delta_{11}w_1^\varepsilon-\delta_{12}w_2^\varepsilon)dx\\
&=-\int_{\mathbb{R}^2}u_1^\varepsilon\left|\nabla(\mu_1\ln u_1^\varepsilon-\delta_{11}w_1^\varepsilon-\delta_{12}w_2^\varepsilon)\right|^2dx.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\int_{\mathbb{R}^2}(\partial_t u_2^\varepsilon)(\mu_2\ln u_2^\varepsilon-\delta_{21}w_1^\varepsilon-\delta_{22}w_2^\varepsilon)dx\\
&=-\int_{\mathbb{R}^2}u_2^\varepsilon\left|\nabla(\mu_2\ln u_2^\varepsilon-\delta_{21}w_1^\varepsilon-\delta_{22}w_2^\varepsilon)\right|^2dx.
\end{aligned}$$

Using the mass conservation property, we have that

$$\frac{d}{dt}\int_{\mathbb{R}^2}\mu_1 u_1^\varepsilon \ln u_1^\varepsilon dx - \delta_{11}\int_{\mathbb{R}^2}(\partial_t u_1^\varepsilon)w_1^\varepsilon dx - \delta_{12}\int_{\mathbb{R}^2}(\partial_t u_1^\varepsilon)w_2^\varepsilon dx \tag{4.32}$$
$$= -\int_{\mathbb{R}^2} u_1^\varepsilon \left|\nabla(\mu_1 \ln u_1^\varepsilon - \delta_{11}w_1^\varepsilon - \delta_{12}w_2^\varepsilon)\right|^2 dx.$$

$$\frac{d}{dt}\int_{\mathbb{R}^2}\mu_2 u_2^\varepsilon \ln u_2^\varepsilon dx - \delta_{21}\int_{\mathbb{R}^2}(\partial_t u_2^\varepsilon)w_1^\varepsilon dx - \delta_{22}\int_{\mathbb{R}^2}(\partial_t u_2^\varepsilon)w_2^\varepsilon dx \tag{4.33}$$
$$= -\int_{\mathbb{R}^2} u_2^\varepsilon \left|\nabla(\mu_2 \ln u_2^\varepsilon - \delta_{21}w_1^\varepsilon - \delta_{22}w_2^\varepsilon)\right|^2 dx.$$

The expression $\frac{1}{\delta_{12}}$*(4.32)+*$\frac{1}{\delta_{21}}$*(4.33) gives*

$$\frac{d}{dt}\left(\frac{\mu_1}{\delta_{12}}\int_{\mathbb{R}^2}u_1^\varepsilon \ln u_1^\varepsilon dx + \frac{\mu_2}{\delta_{21}}\int_{\mathbb{R}^2}u_2^\varepsilon \ln u_2^\varepsilon dx\right) - \frac{\delta_{11}}{\delta_{12}}\int_{\mathbb{R}^2}(\partial_t u_1^\varepsilon)w_1^\varepsilon dx$$
$$-\int_{\mathbb{R}^2}\left((\partial_t u_1^\varepsilon)w_2^\varepsilon + (\partial_t u_2^\varepsilon)w_1^\varepsilon\right)dx - \frac{\delta_{22}}{\delta_{21}}\int_{\mathbb{R}^2}(\partial_t u_2^\varepsilon)w_2^\varepsilon$$
$$= -\frac{1}{\delta_{12}}\int_{\mathbb{R}^2}u_1^\varepsilon\left|\nabla(\mu_1\ln u_1^\varepsilon - \delta_{11}w_1^\varepsilon - \delta_{12}w_2^\varepsilon)\right|^2 dx$$
$$-\frac{1}{\delta_{21}}\int_{\mathbb{R}^2}u_2^\varepsilon\left|\nabla(\mu_2\ln u_2^\varepsilon - \delta_{21}w_1^\varepsilon - \delta_{22}w_2^\varepsilon)\right|^2 dx. \tag{4.34}$$

Notice that

$$\begin{aligned}\int_{\mathbb{R}^2}(\partial_t u_i^\varepsilon)w_i^\varepsilon dx &= \int_{\mathbb{R}^2}\partial_t u_i^\varepsilon\left(\mathbf{K}^\varepsilon * u_i^\varepsilon\right)dx\\ &= \int_{\mathbb{R}^2}\partial_t u_i^\varepsilon(x,t)\left(\frac{1}{4\pi}\int_{\mathbb{R}^2}\ln\frac{1}{|x-y|^2+\varepsilon^2}u_i^\varepsilon(y,t)dy\right)dx\\ &= \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\partial_t u_i^\varepsilon(x,t)\right)u_i^\varepsilon(y,t)\ln\frac{1}{|x-y|^2+\varepsilon^2}dydx\\ &= \frac{1}{8\pi}\frac{d}{dt}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}u_i^\varepsilon(x,t)u_i^\varepsilon(y,t)\ln\frac{1}{|x-y|^2+\varepsilon^2}dydx\\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}u_i^\varepsilon w_i^\varepsilon dx, \end{aligned} \tag{4.35}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^2} ((\partial_t u_1^\varepsilon) w_2^\varepsilon + (\partial_t u_2^\varepsilon) w_1^\varepsilon)\, dx \\
&= \int_{\mathbb{R}^2} ((\partial_t u_1^\varepsilon)(\mathbf{K}^\varepsilon * u_2^\varepsilon) + (\partial_t u_2^\varepsilon)(\mathbf{K}^\varepsilon * u_1^\varepsilon))\, dx \\
&= \int_{\mathbb{R}^2} \partial_t u_1^\varepsilon(x,t) \left(\frac{1}{4\pi} \int_{\mathbb{R}^2} \ln \frac{1}{|x-y|^2+\varepsilon^2} u_2^\varepsilon(y,t) dy \right) dx \\
&+ \int_{\mathbb{R}^2} \partial_t u_2^\varepsilon(x,t) \left(\frac{1}{4\pi} \int_{\mathbb{R}^2} \ln \frac{1}{|x-y|^2+\varepsilon^2} u_1^\varepsilon(y,t) dy \right) dx \\
&= \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\partial_t u_1^\varepsilon(x,t) u_2^\varepsilon(y,t) + \partial_t u_2^\varepsilon(x,t) u_1^\varepsilon(y,t)) \ln \frac{1}{|x-y|^2+\varepsilon^2} dy dx \\
&= \frac{1}{4\pi} \frac{d}{dt} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u_1^\varepsilon(x,t) u_2^\varepsilon(y,t) \log \frac{1}{|x-y|^2+\varepsilon^2} dy dx \\
&= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (u_1^\varepsilon w_2^\varepsilon + u_2^\varepsilon w_1^\varepsilon)\, dx. \qquad (4.36)
\end{aligned}$$

Substituting (4.35) and (4.36) into (4.34), we get

$$\begin{aligned}
&\frac{d}{dt} \left(\frac{\mu_1}{\delta_{12}} \int_{\mathbb{R}^2} u_1^\varepsilon \ln u_1^\varepsilon dx + \frac{\mu_2}{\delta_{21}} \int_{\mathbb{R}^2} u_2^\varepsilon \ln u_2^\varepsilon dx \right. \\
&\left. - \frac{\delta_{11}}{2\delta_{12}} \int_{\mathbb{R}^2} u_1^\varepsilon w_1^\varepsilon dx - \frac{1}{2} \int_{\mathbb{R}^2} (u_1^\varepsilon w_2^\varepsilon + u_2^\varepsilon w_1^\varepsilon)\, dx - \frac{\delta_{22}}{2\delta_{21}} \int_{\mathbb{R}^2} u_2^\varepsilon w_2^\varepsilon dx \right) \\
&= -\frac{1}{\delta_{12}} \int_{\mathbb{R}^2} u_1^\varepsilon \left| \nabla(\mu_1 \ln u_1^\varepsilon - \delta_{11} w_1^\varepsilon - \delta_{12} w_2^\varepsilon) \right|^2 dx \\
&- \frac{1}{\delta_{21}} \int_{\mathbb{R}^2} u_2^\varepsilon \left| \nabla(\mu_2 \ln u_2^\varepsilon - \delta_{21} w_1^\varepsilon - \delta_{22} w_2^\varepsilon) \right|^2 dx.
\end{aligned}$$

Equivalently

$$\begin{aligned}
&\frac{d}{dt} \left(\frac{\mu_1}{\delta_{12}} \int_{\mathbb{R}^2} u_1^\varepsilon \ln u_1^\varepsilon dx + \frac{\mu_2}{\delta_{21}} \int_{\mathbb{R}^2} u_2^\varepsilon \ln u_2^\varepsilon dx \right. \\
&- \frac{1}{2\delta_{12}} \int_{\mathbb{R}^2} u_1^\varepsilon (\delta_{11} w_1^\varepsilon + \delta_{12} w_2^\varepsilon) dx \\
&\left. - \frac{1}{2\delta_{21}} \int_{\mathbb{R}^2} u_2^\varepsilon (\delta_{21} w_1^\varepsilon + \delta_{22} w_2^\varepsilon) dx \right) \\
&= -\frac{1}{\delta_{12}} \int_{\mathbb{R}^2} u_1^\varepsilon \left| \nabla(\mu_1 \ln u_1^\varepsilon - \delta_{11} w_1^\varepsilon - \delta_{12} w_2^\varepsilon) \right|^2 dx \\
&- \frac{1}{\delta_{21}} \int_{\mathbb{R}^2} u_2^\varepsilon \left| \nabla(\mu_2 \ln u_2^\varepsilon - \delta_{21} w_1^\varepsilon - \delta_{22} w_2^\varepsilon) \right|^2 dx. \qquad (4.37)
\end{aligned}$$

Notice that

$$\begin{aligned}
&\begin{pmatrix} \delta_{11} w_1^\varepsilon + \delta_{12} w_2^\varepsilon \\ \delta_{21} w_1^\varepsilon + \delta_{22} w_2^\varepsilon \end{pmatrix} = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix} \begin{pmatrix} w_1^\varepsilon \\ w_2^\varepsilon \end{pmatrix} \\
&= \begin{pmatrix} \chi_{11}\cos\alpha_{11} & \chi_{12}\cos\alpha_{12} \\ \chi_{21}\cos\alpha_{21} & \chi_{22}\cos\alpha_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \begin{pmatrix} v_1^\varepsilon \\ v_2^\varepsilon \end{pmatrix} \\
&= \begin{pmatrix} \chi_{11}\cos\alpha_{11} & \chi_{12}\cos\alpha_{12} \\ \chi_{21}\cos\alpha_{21} & \chi_{22}\cos\alpha_{22} \end{pmatrix} \begin{pmatrix} v_1^\varepsilon \\ v_2^\varepsilon \end{pmatrix} \\
&= \begin{pmatrix} \chi_{11}\cos\alpha_{11} v_1^\varepsilon + \chi_{12}\cos\alpha_{12} v_2^\varepsilon \\ \chi_{21}\cos\alpha_{21} v_1^\varepsilon + \chi_{22}\cos\alpha_{22} v_2^\varepsilon \end{pmatrix}.
\end{aligned}$$

As a consequence, we obtain that the identity (4.37) in terms of the original parameters becomes

$$\frac{d}{dt}\left(\frac{1}{\delta_{12}}\left(\int_{\mathbb{R}^2}\mu_1 u_1^\varepsilon \ln u_1^\varepsilon dx - \frac{1}{2}\int_{\mathbb{R}^2} u_1^\varepsilon(\chi_{11}\cos\alpha_{11}v_1^\varepsilon + \chi_{12}\cos\alpha_{12}v_2^\varepsilon)dx\right)\right.$$
$$\left.+\frac{1}{\delta_{21}}\left(\int_{\mathbb{R}^2}\mu_2 u_2^\varepsilon \ln u_2^\varepsilon dx - \frac{1}{2}\int_{\mathbb{R}^2} u_2^\varepsilon(\chi_{21}\cos\alpha_{21}v_1^\varepsilon + \chi_{22}\cos\alpha_{22}v_2^\varepsilon)dx\right)\right)$$
$$= -\frac{1}{\delta_{12}}\int_{\mathbb{R}^2} u_1^\varepsilon \left|\nabla(\mu_1 \ln u_1^\varepsilon - \chi_{11}\cos\alpha_{11}v_1^\varepsilon - \chi_{12}\cos\alpha_{12}v_2^\varepsilon))\right|^2 dx$$
$$-\frac{1}{\delta_{21}}\int_{\mathbb{R}^2} u_2^\varepsilon \left|\nabla(\mu_2 \ln u_2^\varepsilon - \chi_{21}\cos\alpha_{21}v_1^\varepsilon - \chi_{22}\cos\alpha_{22}v_2^\varepsilon))\right|^2 dx,$$

which is equivalent to (4.23). ■

Boundedness of $\int_{\mathbb{R}^2} u_i^\varepsilon \ln^+ u_i^\varepsilon dx$***.*** Our goal in this step is to show that the positive part of the corresponding entropy functionals, i.e.,

$$S^+\left[u_i^\varepsilon\right](t) := \int_{\mathbb{R}^2} u_i^\varepsilon \ln^+ u_i^\varepsilon dx, \text{ with } i = 1, 2;$$

are bounded on the time interval $(0, T)$ unirformly in ε. Following the technique of [46], we modify the entropy functional within the free energy functional E_ε (4.22), replacing it with a new one that is lower bounded by a constant dependent solely on θ_1 and θ_2.

Let $\delta > 0$ be a any small constant, we introduce the modified free energy E_ε^Γ as follows:

$$E_\varepsilon^\Gamma(t) \tag{4.38}$$
$$:= \frac{1}{\delta_{12}}\left(\int_{\mathbb{R}^2}\mu_1 u_1^\varepsilon \Gamma(u_1^\varepsilon) dx - \frac{1}{2}\int_{\mathbb{R}^2} u_1^\varepsilon(\chi_{11}\cos\alpha_{11}v_1^\varepsilon + \chi_{12}\cos\alpha_{12}v_2^\varepsilon)dx\right)$$
$$+\frac{1}{\delta_{21}}\left(\int_{\mathbb{R}^2}\mu_2 u_2^\varepsilon \Gamma(u_2^\varepsilon) dx - \frac{1}{2}\int_{\mathbb{R}^2} u_2^\varepsilon(\chi_{21}\cos\alpha_{21}v_1^\varepsilon + \chi_{22}\cos\alpha_{22}v_2^\varepsilon)dx\right).$$

where Γ is defined as

$$\Gamma(u) = \begin{cases} \ln u, & u \geq \eta; \\ \ln\eta + \eta^{-1}(u-\eta) - \frac{\eta^{-2}}{2}(u-\eta)^2, & u < \eta. \end{cases} \tag{4.39}$$
$$\eta := \min\left\{1, \frac{\delta\left(\delta_{12}\right)\left(\delta_{21}\right)}{2\left(\delta_{21}\mu_1 + \delta_{12}\mu_2\right)\left(\left(\delta_{11}+\delta_{21}\right)\theta_1 + \left(\delta_{12}+\delta_{22}\right)\theta_2\right)}\right\}.$$

The Γ function is chosen such that it matches with $\ln u$ when $u \geq \eta$, but $\ln(\eta + (u-\eta))$ is replaced by its degree two Taylor expansion centred at η when $u < \eta$. The advantage of this modification is that the function Γ is bounded from below by $\ln\eta - \frac{3}{2}$.

The following theorem shows that, despite the possibility of a slow-growing modified free energy, at most linear growth is possible.

Theorem 37 *Let $(u_1^\varepsilon, u_2^\varepsilon)$ be a solution of system (4.21). Then,*

$$\frac{d}{dt}E_\varepsilon^\Gamma(t) \leq \delta, \tag{4.40}$$

for all $t > 0$. Furthermore, the following quantity is bounded:

$$\int_{u_i^\varepsilon<1} u_i^\varepsilon \Gamma(u_i^\varepsilon)dx \geq \left(-\ln \eta^{-1} - \frac{3}{2}\right)\theta_i, \tag{4.41}$$

where $i = 1, 2$.

Proof. Taking the time derivative of $E_\varepsilon^\Gamma(t)$, we have that

$$\begin{aligned}
&\frac{d}{dt}E_\varepsilon^\Gamma(t) = \frac{d}{dt}\left(\frac{\mu_1}{\delta_{12}}\int_{\mathbb{R}^2} u_1^\varepsilon\Gamma(u_1^\varepsilon)dx + \frac{\mu_2}{\delta_{21}}\int_{\mathbb{R}^2} u_2^\varepsilon\Gamma(u_2^\varepsilon)dx\right)\\
&-\int_{\mathbb{R}^2}\left((\partial_t u_1^\varepsilon)w_2^\varepsilon + (\partial_t u_2^\varepsilon)w_1^\varepsilon\right)dx\\
&-\frac{\delta_{11}}{\delta_{12}}\int_{\mathbb{R}^2}(\partial_t u_1^\varepsilon)w_1^\varepsilon dx - \frac{\delta_{22}}{\delta_{21}}\int_{\mathbb{R}^2}(\partial_t u_2^\varepsilon)w_2^\varepsilon\\
&=\frac{1}{\delta_{12}}\int_{\mathbb{R}^2}(\partial_t u_1^\varepsilon)\left(\mu_1\Gamma(u_1^\varepsilon) - \delta_{11}w_1^\varepsilon - \delta_{12}w_2^\varepsilon\right)dx\\
&+\frac{1}{\delta_{21}}\int_{\mathbb{R}^2}(\partial_t u_2^\varepsilon)\left(\mu_2\Gamma(u_2^\varepsilon) - \delta_{21}w_1^\varepsilon - \delta_{22}w_2^\varepsilon\right)dx\\
&+\frac{\mu_1}{\delta_{12}}\int_{\mathbb{R}^2}u_1^\varepsilon\Gamma'(u_1^\varepsilon)\left(\partial_t u_1^\varepsilon\right)dx + \frac{\mu_2}{\delta_{21}}\int_{\mathbb{R}^2}u_2^\varepsilon\Gamma'(u_2^\varepsilon)\left(\partial_t u_2^\varepsilon\right)dx
\end{aligned}$$

Using that

$$\begin{aligned}
\partial_t u_i^\varepsilon &= \nabla\cdot(u_i^\varepsilon\nabla(\mu_i \ln u_i^\varepsilon - \delta_{i1}w_1^\varepsilon - \delta_{i2}w_2^\varepsilon)\\
&\quad - \nabla\cdot(\delta_{i3}u_i^\varepsilon\nabla^\perp w_1^\varepsilon + \delta_{i4}u_i^\varepsilon\nabla^\perp w_2^\varepsilon),
\end{aligned}$$

for $i = 1, 2$, and Integrating by parts, we get

$$
\begin{aligned}
&\frac{d}{dt} E_\varepsilon^\Gamma(t) \\
&= -\tfrac{1}{\delta_{12}} \int_{\mathbb{R}^2} \nabla(\mu_1 \Gamma(u_1^\varepsilon) - \delta_{11} w_1^\varepsilon - \delta_{12} w_2^\varepsilon) \cdot (u_1^\varepsilon \nabla(\mu_1 \ln u_1^\varepsilon - \delta_{11} w_1^\varepsilon - \delta_{12} w_2^\varepsilon)) dx \\
&+ \tfrac{1}{\delta_{12}} \int_{\mathbb{R}^2} \nabla(\mu_1 \Gamma(u_1^\varepsilon) - \delta_{11} w_1^\varepsilon - \delta_{12} w_2^\varepsilon) \cdot (\delta_{13} u_1^\varepsilon \nabla^\perp w_1^\varepsilon + \delta_{14} u_1^\varepsilon \nabla^\perp w_2^\varepsilon) dx \\
&- \tfrac{\mu_1}{\delta_{12}} \int_{\mathbb{R}^2} \nabla \left(u_1^\varepsilon \Gamma'(u_1^\varepsilon)\right) \cdot (u_1^\varepsilon \nabla(\mu_1 \ln u_1^\varepsilon - \delta_{11} w_1^\varepsilon - \delta_{12} w_2^\varepsilon)) dx \\
&+ \tfrac{\mu_1}{\delta_{12}} \int_{\mathbb{R}^2} \nabla \left(u_1^\varepsilon \Gamma'(u_1^\varepsilon)\right) \cdot (\delta_{13} u_1^\varepsilon \nabla^\perp w_1^\varepsilon + \delta_{14} u_1^\varepsilon \nabla^\perp w_2^\varepsilon) dx \\
&- \tfrac{1}{\delta_{21}} \int_{\mathbb{R}^2} \nabla(\mu_2 \Gamma(u_2^\varepsilon) - \delta_{21} w_1^\varepsilon - \delta_{22} w_2^\varepsilon) \cdot (u_2^\varepsilon \nabla(\mu_2 \ln u_2^\varepsilon - \delta_{21} w_1^\varepsilon - \delta_{22} w_2^\varepsilon)) dx \\
&+ \tfrac{1}{\delta_{21}} \int_{\mathbb{R}^2} \nabla(\mu_2 \Gamma(u_2^\varepsilon) - \delta_{21} w_1^\varepsilon - \delta_{22} w_2^\varepsilon) \cdot (\delta_{23} u_2^\varepsilon \nabla^\perp w_1^\varepsilon + \delta_{24} u_2^\varepsilon \nabla^\perp w_2^\varepsilon) dx \\
&- \tfrac{\mu_2}{\delta_{21}} \int_{\mathbb{R}^2} \nabla \left(u_2^\varepsilon \Gamma'(u_2^\varepsilon)\right) \cdot (u_2^\varepsilon \nabla(\mu_2 \ln u_2^\varepsilon - \delta_{21} w_1^\varepsilon - \delta_{22} w_2^\varepsilon))\, dx \\
&+ \tfrac{\mu_2}{\delta_{21}} \int_{\mathbb{R}^2} \nabla \left(u_2^\varepsilon \Gamma'(u_2^\varepsilon)\right) \cdot \left(\delta_{23} u_2^\varepsilon \nabla^\perp w_1^\varepsilon + \delta_{24} u_2^\varepsilon \nabla^\perp w_2^\varepsilon\right) dx \\
&=: \sum_{i=1}^{8} T_i
\end{aligned}
$$

To estimate the second term T_2 and the fourth term T_4, we define the following functions:

$$
\xi(u) = \int_0^u s\Gamma'(s) ds, \qquad \varsigma(u) = \int_0^u s^2 \Gamma''(s) ds.
$$

Using that $\nabla \cdot \nabla^\perp w_i^\varepsilon = \nabla w_i^\varepsilon \cdot \nabla^\perp w_i^\varepsilon = 0, i = 1, 2$, $\nabla w_2^\varepsilon \cdot \nabla^\perp w_1^\varepsilon = -\nabla w_1^\varepsilon \cdot \nabla^\perp w_2^\varepsilon$ and (4.31), we have that

$$
\begin{aligned}
T_2 &= \frac{\mu_1 \delta_{13}}{\delta_{12}} \int_{\mathbb{R}^2} \nabla \xi(u_1^\varepsilon) \cdot \nabla^\perp w_1^\varepsilon dx + \frac{\mu_1 \delta_{14}}{\delta_{12}} \int_{\mathbb{R}^2} \nabla \xi(u_1^\varepsilon) \cdot \nabla^\perp w_2^\varepsilon dx \\
&- \frac{\delta_{11} \delta_{13}}{\delta_{12}} \int_{\mathbb{R}^2} u_1^\varepsilon \nabla w_1^\varepsilon \cdot \nabla^\perp w_1^\varepsilon dx - \frac{\delta_{11} \delta_{14}}{\delta_{12}} \int_{\mathbb{R}^2} u_1^\varepsilon \nabla w_1^\varepsilon \cdot \nabla^\perp w_2^\varepsilon dx \\
&- \delta_{13} \int_{\mathbb{R}^2} u_1^\varepsilon \nabla w_2^\varepsilon \cdot \nabla^\perp w_1^\varepsilon dx - \delta_{14} \int_{\mathbb{R}^2} u_1^\varepsilon \nabla w_2^\varepsilon \cdot \nabla^\perp w_2^\varepsilon dx \\
&= -\frac{\mu_1 \delta_{13}}{\delta_{12}} \int_{\mathbb{R}^2} \xi(u_1^\varepsilon) \nabla \cdot \nabla^\perp w_1^\varepsilon dx - \frac{\mu_1 \delta_{14}}{\delta_{12}} \int_{\mathbb{R}^2} \xi(u_1^\varepsilon) \nabla \cdot \nabla^\perp w_2^\varepsilon dx \\
&- \left(\frac{\delta_{11} \delta_{14} - \delta_{12} \delta_{13}}{\delta_{12}} \right) \int_{\mathbb{R}^2} u_1^\varepsilon \nabla w_1^\varepsilon \cdot \nabla^\perp w_2^\varepsilon dx \\
&= 0.
\end{aligned}
$$

and

$$
\begin{aligned}
T_4 &= \frac{\mu_1}{\delta_{12}} \int_{\mathbb{R}^2} u_1^\varepsilon \nabla \left(u_1^\varepsilon \Gamma'(u_1^\varepsilon)\right) \cdot \left(\delta_{13} \nabla^\perp w_1^\varepsilon + \delta_{14} \nabla^\perp w_2^\varepsilon\right) dx \\
&= \frac{\mu_1}{\delta_{12}} \int_{\mathbb{R}^2} \nabla \xi(u_1^\varepsilon) \cdot \left(\delta_{13} \nabla^\perp w_1^\varepsilon + \delta_{14} \nabla^\perp w_2^\varepsilon\right) dx \\
&+ \frac{\mu_1}{\delta_{12}} \int_{\mathbb{R}^2} \nabla \varsigma(u_1^\varepsilon) \cdot \left(\delta_{13} \nabla^\perp w_1^\varepsilon + \delta_{14} v w_2^\varepsilon\right) dx \\
&= \frac{\mu_1}{\delta_{12}} \int_{\mathbb{R}^2} \xi(u_1^\varepsilon) \nabla \cdot \left(\delta_{13} \nabla^\perp w_1^\varepsilon + \delta_{14} \nabla^\perp w_2^\varepsilon\right) dx \\
&+ \frac{\mu_1}{\delta_{12}} \int_{\mathbb{R}^2} \varsigma(u_1^\varepsilon) \nabla \cdot \left(\delta_{13} \nabla^\perp w_1^\varepsilon + \delta_{14} \nabla^\perp w_2^\varepsilon\right) dx \\
&= 0.
\end{aligned}
$$

Similarly, we have that $T_6 = T_8 = 0$. On the other hand, simple computations show that

$$
\Gamma'(u) = 2\eta^{-1} - \eta^{-2} u \quad \text{for} \quad u \leq \eta.
$$

Now we estimate the terms $T_1 + T_3$ as follows:

$$
\begin{aligned}
T_1 + T_3 = &- \frac{1}{\delta_{12}} \int_{u_1^\varepsilon \geq \eta} u_1^\varepsilon \left|\nabla(\mu_1 \ln u_1^\varepsilon - \delta_{11} w_1^\varepsilon - \delta_{12} w_2^\varepsilon)\right|^2 dx \\
&- \frac{\mu_1^2}{\delta_{12}} \int_{u_1^\varepsilon < \eta} (4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon) \left|\nabla u_1^\varepsilon\right|^2 dx \\
&+ \frac{\mu_1}{\delta_{12}} \int_{u_1^\varepsilon < \eta} ((4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon) u_1^\varepsilon \nabla u_1^\varepsilon \cdot (\delta_{11} \nabla w_1^\varepsilon + \delta_{12} \nabla w_2^\varepsilon)\, dx \\
&- \frac{1}{\delta_{12}} \int_{u_1^\varepsilon < \eta} u_1^\varepsilon \left|\delta_{11} \nabla w_1^\varepsilon + \delta_{12} \nabla w_2^\varepsilon\right|^2 \\
&+ \frac{\mu_1}{\delta_{12}} \int_{u_1^\varepsilon < \eta} \nabla u_1^\varepsilon \cdot (\delta_{11} \nabla w_1^\varepsilon + \delta_{12} \nabla w_2^\varepsilon)\, dx.
\end{aligned}
$$

Using the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
T_1 + T_3 \leq &- \frac{1}{\delta_{12}} \int_{u_1^\varepsilon \geq \eta} u_1^\varepsilon \left|\nabla(\mu_1 \ln u_1^\varepsilon - \delta_{11} w_1^\varepsilon - \delta_{12} w_2^\varepsilon)\right|^2 dx \\
&- \frac{\mu_1^2}{\delta_{12}} \int_{u_1^\varepsilon < \eta} (4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon) \left|\nabla u_1^\varepsilon\right|^2 dx \\
&+ \frac{\mu_1}{\delta_{12}} \int_{u_1^\varepsilon < \eta} ((4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon) u_1^\varepsilon \left|\nabla u_1^\varepsilon\right| \left|\delta_{11} \nabla w_1^\varepsilon + \delta_{12} \nabla w_2^\varepsilon\right| dx \\
&- \frac{1}{\delta_{12}} \int_{u_1^\varepsilon < \eta} u_1^\varepsilon \left|\delta_{11} \nabla w_1^\varepsilon + \delta_{12} \nabla w_2^\varepsilon\right|^2 \\
&+ \frac{\mu_1}{\delta_{12}} \int_{u_1^\varepsilon < \eta} \nabla u_1^\varepsilon \cdot (\delta_{11} \nabla w_1^\varepsilon + \delta_{12} \nabla w_2^\varepsilon)\, dx.
\end{aligned}
$$

Notice that

$$
\sup_{0 \leq u \leq \eta} \sqrt{(4\eta^{-1} - 3\eta^{-2} u)\, u} \leq \frac{2\sqrt{3}}{3} < 2,
$$

which implies,

$$
\begin{aligned}
T_1 + T_3 \le & -\frac{1}{\delta_{12}} \int_{u_1^\varepsilon \ge \eta} u_1^\varepsilon \left| \nabla(\mu_1 \ln u_1^\varepsilon - \delta_{11} w_1^\varepsilon - \delta_{12} w_2^\varepsilon) \right|^2 dx \\
& - \frac{\mu_1^2}{\delta_{12}} \int_{u_1^\varepsilon < \eta} (4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon) \left| \nabla u_1^\varepsilon \right|^2 dx \\
& + \frac{2\sqrt{3}\mu_1}{3\delta_{12}} \int_{u_1^\varepsilon < \eta} \sqrt{(4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon) u_1^\varepsilon} \left| \nabla u_1^\varepsilon \right| \left| \delta_{11} \nabla w_1^\varepsilon + \delta_{12} \nabla w_2^\varepsilon \right| dx \\
& - \frac{1}{\delta_{12}} \int_{u_1^\varepsilon < \eta} u_1^\varepsilon \left| \delta_{11} \nabla w_1^\varepsilon + \delta_{12} \nabla w_2^\varepsilon \right|^2 \\
& + \frac{\mu_1}{\delta_{12}} \int_{u_1^\varepsilon < \eta} \nabla u_1^\varepsilon \cdot (\delta_{11} \nabla w_1^\varepsilon + \delta_{12} \nabla w_2^\varepsilon) \, dx.
\end{aligned}
$$

Completing a square using the 2nd, 3rd, 4th terms in the last line, we obtain that

$$
\begin{aligned}
T_1 + T_3 \le & -\frac{1}{\delta_{12}} \int_{u_1^\varepsilon \ge \eta} u_1^\varepsilon \left| \nabla(\mu_1 \ln u_1^\varepsilon - \delta_{11} w_1^\varepsilon - \delta_{12} w_2^\varepsilon) \right|^2 dx \\
& - \frac{2\mu_1^2}{3\delta_{12}} \int_{u_1^\varepsilon < \eta} (4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon) \left| \nabla u_1^\varepsilon \right|^2 dx \\
& - \frac{1}{\delta_{12}} \int_{u_1^\varepsilon < \eta} \left(\frac{\mu_1 \sqrt{3}}{3} \sqrt{(4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon)} \left| \nabla u_1^\varepsilon \right| - \sqrt{u_1^\varepsilon} \left| \delta_{11} \nabla w_1^\varepsilon + \delta_{12} \nabla w_2^\varepsilon \right| \right)^2 dx \\
& + \frac{\mu_1}{\delta_{12}} \int_{u_1^\varepsilon < \eta} \nabla u_1^\varepsilon \cdot (\delta_{11} \nabla w_1^\varepsilon + \delta_{12} \nabla w_2^\varepsilon) \, dx.
\end{aligned}
$$

Next, we have that

$$
\begin{aligned}
& \frac{\mu_1}{\delta_{12}} \int_{u_1^\varepsilon < \eta} \nabla u_1^\varepsilon \cdot (\delta_{11} \nabla w_1^\varepsilon + \delta_{12} \nabla w_2^\varepsilon) \, dx \\
& = \frac{\mu_1}{\delta_{12}} \int_{\mathbb{R}^2} \nabla(\min\{u_1^\varepsilon, \eta\}) \cdot (\delta_{11} \nabla w_1^\varepsilon + \delta_{12} \nabla w_2^\varepsilon) \, dx \\
& = \frac{\mu_1}{\delta_{12}} \int_{\mathbb{R}^2} \min\{u_1^\varepsilon, \eta\} \left(- \left(\delta_{11} \Delta w_1^\varepsilon + \delta_{12} \Delta w_2^\varepsilon \right) \right) dx \\
& \le \frac{\mu_1}{\delta_{12}} \eta \int_{\mathbb{R}^2} \left(-\triangle \mathbf{K}^\varepsilon * (\delta_{11} u_1^\varepsilon + \delta_{12} u_2^\varepsilon) \right) dx \\
& \le \frac{\mu_1}{\delta_{12}} \eta \left\| \triangle \mathbf{K}^\varepsilon \right\|_{L^1(\mathbb{R}^2)} (\delta_{11} \theta_1 + \delta_{12} \theta_2) \le \frac{\delta}{2}.
\end{aligned}
$$

Here we have applied that $\min\{u_1^\varepsilon, \eta\} \in W^{1,p}(\mathbb{R}^2)$ and $\nabla(\min\{u_1^\varepsilon, \eta\}) = 1_{\{u<\eta\}} \nabla u$ a.e. since $u_1^\varepsilon \in W^{1,p}(\mathbb{R}^2)$, for $1 \le p < \infty$ (By Lemma 16). Moreover, to justify the integration by parts, we can use a sequence of functions $\psi_n \in C_0^\infty(\mathbb{R}^2)$ such that $\psi_n \to \min\{u_1^\varepsilon, \eta\}$ in $W^{1,4/3}(\mathbb{R}^2)$.We also notice that $\nabla w_i^\varepsilon \in W^{1,p}(\mathbb{R}^2)^2, i = 1, 2$, for $p \in (2, \infty)$. Therefore

$$
\int_{\mathbb{R}^2} \nabla \psi_n \cdot \nabla w_i^\varepsilon dx = - \int_{\mathbb{R}^2} \psi_n \Delta w_i^\varepsilon dx. \tag{4.42}
$$

Now, we can pass to the limit in (4.42) when $n \to \infty$, since

$$\int_{\mathbb{R}^2} \left(\nabla\psi_n - \nabla\left(\min\{u_1^\varepsilon, \eta\}\right)\right) \cdot \nabla w_i^\varepsilon dx$$
$$\leq \|\nabla\psi_n - \nabla\left(\min\{u_1^\varepsilon, \eta\}\right)\|_{L^{4/3}(\mathbb{R}^2)} \||\nabla w_i^\varepsilon|\|_{L^4(\mathbb{R}^2)} \to 0,$$

and

$$\int_{\mathbb{R}^2} \left(\psi_n - \min\{u_1^\varepsilon, \eta\}\right) \Delta w_i^\varepsilon dx$$
$$\leq \|\psi_n - \min\{u_1^\varepsilon, \eta\}\|_{L^{4/3}(\mathbb{R}^2)} \|\Delta w_i^\varepsilon\|_{L^4(\mathbb{R}^2)} \to 0.$$

Thus,

$$\int_{\mathbb{R}^2} \nabla\left(\min\{u_1^\varepsilon, \eta\}\right) \cdot \nabla w_i^\varepsilon dx = -\int_{\mathbb{R}^2} \min\{u_1^\varepsilon, \eta\} \Delta w_i^\varepsilon dx.$$

In summary

$$T_1 + T_3 \leq \frac{\delta}{2} - \frac{1}{\delta_{12}} \int_{u_1^\varepsilon \geq \eta} u_1^\varepsilon \left|\nabla(\mu_1 \ln u_1^\varepsilon - \delta_{11} w_1^\varepsilon - \delta_{12} w_2^\varepsilon)\right|^2 dx$$
$$- \frac{2\mu_1^2}{3\delta_{12}} \int_{u_1^\varepsilon < \eta} (4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon) \left|\nabla u_1^\varepsilon\right|^2 dx$$
$$- \frac{1}{\delta_{12}} \int_{u_1^\varepsilon < \eta} \left(\frac{\mu_1 \sqrt{3}}{3} \sqrt{(4\eta^{-1} - 3\eta^{-2} u_1^\varepsilon)} \left|\nabla u_1^\varepsilon\right| - \sqrt{u_1^\varepsilon} \left|\delta_{11} \nabla w_1^\varepsilon + \delta_{12} \nabla w_2^\varepsilon\right| \right)^2 dx$$
$$\leq \frac{\delta}{2}.$$

Similarly, we obtain that $T_5 + T_7 \leq \frac{\delta}{2}$.Therefore, the estimate (4.40) follows.

Estimate (4.41) follows from the fact that the function Γ is bounded from below by $-\ln \eta^{-1} - \frac{3}{2} \leq 0$. Indeed,

$$\int_{u_i^\varepsilon < 1} u_i^\varepsilon \Gamma(u_i^\varepsilon) dx \geq \left(-\ln \eta^{-1} - \frac{3}{2}\right) \int_{u_i^\varepsilon < 1} u_i^\varepsilon dx \geq \left(-\ln \eta^{-1} - \frac{3}{2}\right) \theta_i.$$

■

In order to apply the Logarithmic HLS inequality for systems (See Theorem 18) in the region of the plane $\theta_1\theta_2$ defined by (4.14), we introduce the next technical lemma

Lemma 38 *Let us assume that* (θ_1, θ_2) *satisfies*

$$\delta_{11}\theta_1 < 8\pi\mu_1, \ \delta_{22}\theta_2 < 8\pi\mu_2,$$
$$\text{and } \frac{8\pi\mu_1}{\delta_{12}}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right) > 0. \tag{4.43}$$

then there are constants $b_1 \in (\delta_{12}, \infty)$ *and* $b_2 \in (\delta_{21}, \infty)$ *depending on the parameters* $\theta_i, \mu_i, \chi_{ij}, a_{ij}, \alpha_{ij}$ *with* $i, j = 1, 2$ *such that*

$$\delta_{11}\theta_1 < \delta_{12}\frac{8\pi\mu_1}{b_1}, \ \delta_{22}\theta_2 < \delta_{21}\frac{8\pi\mu_2}{b_2}, \tag{4.44}$$

and

$$\frac{8\pi\mu_1}{b_1}\theta_1 + \frac{8\pi\mu_2}{b_2}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right) = 0. \tag{4.45}$$

Proof. Assume $\delta_{11}, \delta_{22} > 0$[1]. By hypothesis $\theta_1 \in \left(0, \frac{8\pi\mu_1}{\delta_{11}}\right)$ and it is clear that $\theta_1 \in \left(0, \theta_1 + \frac{\theta_2}{2}\frac{\delta_{12}}{\delta_{11}}\right)$.

Then, we have that $\theta_1 \in \left(0, \frac{8\pi\mu_1}{\delta_{11}}\right) \cap \left(0, \theta_1 + \frac{\theta_2}{2}\frac{\delta_{12}}{\delta_{11}}\right)$ which implies the existence of a constant $s_1 > 0$ satisfying

$$\theta_1 < \frac{\delta_{12}}{\delta_{11}}\frac{8\pi\mu_1}{\delta_{12}+s_1} < \theta_1 + \frac{\theta_2}{2}\frac{\delta_{12}}{\delta_{11}}. \tag{4.46}$$

Similarly, $\theta_2 \in \left(0, \frac{8\pi\mu_2}{\delta_{22}}\right) \cap \left(0, \theta_2 + \frac{\theta_1}{2}\frac{\delta_{21}}{\delta_{22}}\right)$ implies the existence of a constant $s_2 > 0$ satisfying

$$\theta_2 < \frac{\delta_{21}}{\delta_{22}}\frac{8\pi\mu_2}{\delta_{21}+s_2} < \theta_2 + \frac{\theta_1}{2}\frac{\delta_{21}}{\delta_{22}}. \tag{4.47}$$

Let us define the function $f : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ by

$$f(x,y) := \frac{8\pi\mu_1}{x}\theta_1 + \frac{8\pi\mu_2}{y}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right)$$

Taking $x = \delta_{12} + s_1$ and $y = \delta_{21} + s_2$, we obtain

$$\begin{aligned}
&f(\delta_{12}+s_1, \delta_{21}+s_2) \\
&= \frac{8\pi\mu_1}{\delta_{12}+s_1}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}+s_2}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right) \\
&= \left(\frac{8\pi\mu_1}{\delta_{12}+s_1} - \frac{\delta_{11}}{\delta_{12}}\theta_1\right)\theta_1 + \left(\frac{8\pi\mu_2}{\delta_{21}+s_2} - \frac{\delta_{22}}{\delta_{21}}\theta_2\right)\theta_2 - 2\theta_1\theta_2 \\
&= \frac{\delta_{11}}{\delta_{12}}\left(\frac{\delta_{12}}{\delta_{11}}\frac{8\pi\mu_1}{\delta_{12}+s_1} - \theta_1\right)\theta_1 + \frac{\delta_{22}}{\delta_{21}}\left(\frac{\delta_{21}}{\delta_{22}}\frac{8\pi\mu_2}{\delta_{21}+s_2} - \theta_2\right)\theta_2 - 2\theta_1\theta_2
\end{aligned}$$

Then, an application of the right side of (4.46) and (4.47), respectively give us

$$f(\delta_{12}+s_1, \delta_{21}+s_2) < \frac{\theta_1\theta_2}{2} + \frac{\theta_1\theta_2}{2} - 2\theta_1\theta_2 = -\theta_1\theta_2 < 0 \tag{4.48}$$

Let us now define the function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$ by

$$g(\tau) := f(\delta_{12} + s_1\tau, \delta_{21} + s_2\tau).$$

Note that

$$g(0) = \frac{8\pi\mu_1}{\delta_{12}}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right) > 0$$

and from (4.48) we get $g(1) < 0$. Thus for some $\tau^* \in (0,1)$ it holds $g(\tau^*) = 0$. Let us call

$$b_1 := \delta_{12} + s_1\tau^* \text{ and } b := \delta_{21} + s_2\tau^*$$

[1]Note that if $\delta_{ii} = 0$, we have that there is a constant $s_i > 0$ such that

$$\left(\frac{8\pi\mu_i}{\delta_{i2}+s_i} - \frac{\delta_{ii}}{\delta_{i2}}\theta_i\right)\theta_i = \frac{8\pi\mu_i\theta_i}{\delta_{i2}+s_i} < \frac{\theta_1\theta_2}{2}.$$

From the left side of (4.46)

$$\theta_1 < \frac{\delta_{12}}{\delta_{11}}\frac{8\pi\mu_1}{\delta_{12}+s_1} < \frac{\delta_{12}}{\delta_{11}}\frac{8\pi\mu_1}{\delta_{12}+s_1\tau^*} = \frac{\delta_{12}}{\delta_{11}}\frac{8\pi\mu_1}{b_1}$$

Similarly from (4.47)

$$\theta_2 < \frac{\delta_{21}}{\delta_{22}}\frac{8\pi\mu_2}{b_2}$$

The inequality (4.45) follows from $g(\tau^*) = 0$. ■

Theorem 39 *Consider a non-negative solution of (4.21) such that* $u_i^\varepsilon \ln(1+|x|^2), u_i^\varepsilon \ln u_i^\varepsilon \in L^\infty_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$ *for* $i = 1, 2$. *If* (θ_1, θ_2) *satisfies*

$$\begin{gathered}\delta_{11}\theta_1 < 8\pi\mu_1,\ \ \delta_{22}\theta_2 < 8\pi\mu_2, \\ \text{and } \frac{8\pi\mu_1}{\delta_{12}}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right) > 0.\end{gathered} \tag{4.49}$$

then for any real $\delta > 0$, *there exists a constant* $C_{S^+} := C(\delta)$ *such that*

$$\int_{\mathbb{R}^2} u_i^\varepsilon(x,t)\ln^+ u_i^\varepsilon(x,t)dx \le C_{S^+} + \delta T,\ \text{ for any } t \in [0,T], \tag{4.50}$$

where $i = 1, 2$.

Proof. From (4.40) we have that

$$E_\varepsilon^\Gamma(t) \le E^\Gamma(0) + \delta t,\ \text{ for any } t > 0.$$

Hence, we estimate the following

$$\begin{aligned}&\frac{\mu_1}{\delta_{12}}\int_{\mathbb{R}^2} u_1^\varepsilon(x,t)\Gamma(u_1^\varepsilon(x,t))dx + \frac{\mu_2}{\delta_{21}}\int_{\mathbb{R}^2} u_2^\varepsilon(x,t)\Gamma(u_2^\varepsilon(x,t))dx \\ &\le E^\Gamma(0) + \delta t - \frac{1}{4\pi}\frac{\delta_{11}}{\delta_{12}}\int_{\mathbb{R}^2\times\mathbb{R}^2} u_1^\varepsilon(x,t)u_1^\varepsilon(y,t)\ln|x-y|dxdy \\ &- \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} u_1^\varepsilon(x,t)u_2^\varepsilon(y,t)\ln|x-y|dxdy \\ &- \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} u_2^\varepsilon(x,t)u_1^\varepsilon(y,t)\ln|x-y|dxdy \\ &- \frac{1}{4\pi}\frac{\delta_{22}}{\delta_{21}}\int_{\mathbb{R}^2\times\mathbb{R}^2} u_2^\varepsilon(x,t)u_2^\varepsilon(y,t)\ln|x-y|dxdy.\end{aligned}$$

Applying the definition of Γ (4.39) and (4.41) we get

$$\begin{aligned}&\frac{\mu_1}{\delta_{12}}\int_{\mathbb{R}^2} u_1^\varepsilon(x,t)\ln^+ u_1^\varepsilon(x,t)dx + \frac{\mu_2}{\delta_{21}}\int_{\mathbb{R}^2} u_2^\varepsilon(x,t)\ln^+ u_2^\varepsilon(x,t)dx \\ &\le E^\Gamma(0) + \delta t + \frac{\mu_1\theta_1}{\delta_{12}}\left(\ln\eta^{-1} + \frac{3}{2}\right) + \frac{\mu_2\theta_2}{\delta_{21}}\left(\ln\eta^{-1} + \frac{3}{2}\right) \\ &- \frac{1}{4\pi}\frac{\delta_{11}}{\delta_{12}}\int_{\mathbb{R}^2\times\mathbb{R}^2} u_1^\varepsilon(x,t)u_1^\varepsilon(y,t)\ln|x-y|dxdy \\ &- \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} u_1^\varepsilon(x,t)u_2^\varepsilon(y,t)\ln|x-y|dxdy \\ &- \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} u_2^\varepsilon(x,t)u_1^\varepsilon(y,t)\ln|x-y|dxdy \\ &- \frac{1}{4\pi}\frac{\delta_{22}}{\delta_{21}}\int_{\mathbb{R}^2\times\mathbb{R}^2} u_2^\varepsilon(x,t)u_2^\varepsilon(y,t)\ln|x-y|dxdy.\end{aligned}$$

In the next step, positive parameters b_1 and b_2 are introduced in the following way

$$\begin{aligned}
&\frac{\mu_1}{\delta_{12}}\int_{\mathbb{R}^2} u_1^\varepsilon(x,t)\ln^+ u_1^\varepsilon(x,t)dx + \frac{\mu_2}{\delta_{21}}\int_{\mathbb{R}^2} u_2^\varepsilon(x,t)\ln^+ u_2^\varepsilon(x,t)dx \\
&\le E^\Gamma(0) + \delta t + \frac{\mu_1\theta_1}{\delta_{12}}\left(\ln\eta^{-1} + \frac{3}{2}\right) + \frac{\mu_2\theta_2}{\delta_{21}}\left(\ln\eta^{-1} + \frac{3}{2}\right) \\
&- \frac{b_1^2}{4\pi\mu_1^2}\frac{\delta_{11}}{\delta_{12}}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mu_1 u_1^\varepsilon(x,t)}{b_1}\frac{\mu_1 u_1^\varepsilon(y,t)}{b_1}\ln|x-y|dxdy \\
&- \frac{b_1 b_2}{4\pi\mu_1\mu_2}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mu_1 u_1^\varepsilon(x,t)}{b_1}\frac{\mu_2 u_2^\varepsilon(y,t)}{b_2}\ln|x-y|dxdy \\
&- \frac{b_1 b_2}{4\pi\mu_1\mu_2}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mu_2 u_2^\varepsilon(x,t)}{b_2}\frac{\mu_1 u_1^\varepsilon(y,t)}{b_1}\ln|x-y|dxdy \\
&- \frac{b_2^2}{4\pi\mu_2^2}\frac{\delta_{22}}{\delta_{21}}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mu_2 u_2^\varepsilon(x,t)}{b_2}\frac{\mu_2 u_2^\varepsilon(y,t)}{b_2}\ln|x-y|dxdy. \qquad (4.51)
\end{aligned}$$

Now, we can apply (18) to the functions $\frac{\mu_1 u_1^\varepsilon}{b_1}$ and $\frac{\mu_2 u_2^\varepsilon}{b_2}$ in right side of (4.51) getting that

$$\begin{aligned}
&\frac{\mu_1}{\delta_{12}}\int_{\mathbb{R}^2} u_1^\varepsilon(x,t)\ln^+ u_1^\varepsilon(x,t)dx + \frac{\mu_2}{\delta_{21}}\int_{\mathbb{R}^2} u_2^\varepsilon(x,t)\ln^+ u_2^\varepsilon(x,t)dx \\
&\le E^\Gamma(0) + \delta t + \frac{\mu_1\theta_1}{\delta_{12}}\left(\ln\eta^{-1} + \frac{3}{2}\right) + \frac{\mu_2\theta_2}{\delta_{21}}\left(\ln\eta^{-1} + \frac{3}{2}\right) - C_{HLS} \\
&+ \int_{\mathbb{R}^2}\frac{\mu_1 u_1^\varepsilon(x,t)}{b_1}\ln\left(\frac{\mu_1 u_1^\varepsilon(x,t)}{b_1}\right)dx + \int_{\mathbb{R}^2}\frac{\mu_2 u_2^\varepsilon(x,t)}{b_2}\ln\left(\frac{\mu_2 u_2^\varepsilon(x,t)}{b_2}\right)dx,
\end{aligned}$$

where the conditions for the existence of the constant C_{HLS} given by Logarithmic HLS inequality for systems are

$$\begin{aligned}
&\delta_{11}\theta_1 \le \delta_{12}\frac{8\pi\mu_1}{b_1},\ \delta_{22}\theta_2 \le \delta_{21}\frac{8\pi\mu_2}{b_2}, \\
\text{and } &\frac{8\pi\mu_1}{b_1}\theta_1 + \frac{8\pi\mu_2}{b_2}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right) = 0
\end{aligned} \qquad (4.52)$$

In conclusion we have proved that the conditions (4.52) implies

$$\begin{aligned}
&\mu_1\left(\frac{1}{\delta_{12}} - \frac{1}{b_1}\right)\int_{\mathbb{R}^2} u_1^\varepsilon(x,t)\ln^+ u_1^\varepsilon(x,t)dx \\
&+ \mu_2\left(\frac{1}{\delta_{21}} - \frac{1}{b_2}\right)\int_{\mathbb{R}^2} u_2^\varepsilon(x,t)\ln^+ u_2^\varepsilon(x,t)dx \\
&\le E^\Gamma(0) + \delta T - C_{HLS} + \left(\frac{\mu_1\theta_1}{\delta_{12}} + \frac{\mu_2\theta_2}{\delta_{21}}\right)\left(\ln\eta^{-1} + \frac{3}{2}\right) \\
&+ \frac{\mu_1\theta_1}{b_1}\ln\left(\frac{\mu_1}{b_1}\right) + \frac{\mu_2\theta_2}{b_2}\ln\left(\frac{\mu_2}{b_2}\right). \qquad (4.53)
\end{aligned}$$

Note that each of the coefficients of the positive part of the entropy functionals in (4.53) are positive providing $b_1 \in (\delta_{12}, \infty)$ and $b_2 \in (\delta_{21}, \infty)$. Then, we have that $\int u_i^\varepsilon \ln^+ u_i^\varepsilon dx$ are bounded below for $i = 1, 2$. The Lemma 38 gives us that the estimate (4.50) holds for the region (4.49). ■

Boundedness of L^p ***norm for*** $1 < p < \infty$**.** The purpose of this step is to obtain estimates of the L^p-norms for $1 < p < \infty$ of the variables u_1^ε and u_2^ε independent of the parameter ε.

Proposition 40 *Assume that* $u_{10}, u_{20} \in L^1(\mathbb{R}^2, \ln(1+|x|^2)dx)$*,* $u_{10} \ln u_{10}$*,* $u_{20} \ln u_{20} \in L^1(\mathbb{R}^2, dx)$ *and* (θ_1, θ_2) *satisfies*

$$\delta_{11}\theta_1 < 8\pi\mu_1,\ \ \delta_{22}\theta_2 < 8\pi\mu_2,$$
$$\text{and } \frac{8\pi\mu_1}{\delta_{12}}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right) > 0.$$

If u_{10}, u_{20} *are bounded in* $L^p(\mathbb{R}^2)$ *for some* $p \in (1, \infty)$*, .then any solution* $(u_1^\varepsilon, u_2^\varepsilon)$ *of (4.21) is bounded in* $L^\infty_{loc}(\mathbb{R}^+, L^p(\mathbb{R}^2))$*.*

Proof. We decompose u_i^ε as follows:

$$u_i^\varepsilon = (u_i^\varepsilon - K)_+ + \min\{u_i^\varepsilon, K\},\ \ K > 1.$$

Note that the function $\min\{u_i^\varepsilon, K\} \in L^p(\mathbb{R}^2)$ is bounded in L^p by $K^{p-1}\theta_i$. Indeed,

$$\int_{\mathbb{R}^2} (\min\{u_i^\varepsilon, K\})^p \, dx \le K^{p-1} \int_{\mathbb{R}^2} u_i^\varepsilon dx = K^{p-1}\theta_i.$$

Then, it is enough to estimate the L^p norm of $(u_i^\varepsilon - K)_+$. For this purpose, we define first

$$M_i(K) := \int_{\mathbb{R}^2} (u_i^\varepsilon - K)_+ \, dx.$$

Using the fact that $u_i^\varepsilon \ln^+ u_i^\varepsilon$ is bounded in $L^\infty(\mathbb{R}^+_{Loc}, L^1(\mathbb{R}^2))$, we can estimate $M_i(K)$ by

$$M_i(K) \le \frac{1}{\ln K} \int_{\mathbb{R}^2} u_i^\varepsilon \ln^+ u_i^\varepsilon dx.$$

and choose it arbitrarily small on any given time interval $(0, T)$.

Multiplying the first equation of system (4.21) by $(u_1^\varepsilon - K)_+^{p-1}$ and integrating over $\mathbb{R}^2$, we get

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^p \, dx = \mu_1 \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} \triangle u_1^\varepsilon dx \\
&- \chi_{11} \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} \nabla \cdot (u_1^\varepsilon A_{11} \nabla v_1^\varepsilon) dx \\
&- \chi_{12} \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} \nabla \cdot (u_1^\varepsilon A_{12} \nabla v_2^\varepsilon) dx \\
&= \mu_1 \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} \triangle u_1^\varepsilon dx - \chi_{11}\cos\alpha_{11} \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} \nabla \cdot (u_1^\varepsilon \nabla v_1^\varepsilon) dx \\
&- \chi_{11}\sin\alpha_{11} \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} \nabla \cdot (u_1^\varepsilon \nabla^\perp v_1^\varepsilon) dx \\
&- \chi_{12}\cos\alpha_{12} \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} \nabla \cdot (u_1^\varepsilon \nabla v_2^\varepsilon) dx \\
&- \chi_{12}\sin\alpha_{12} \int_{\mathbb{R}^2} (u_1^\varepsilon - K)_+^{p-1} \nabla \cdot (u_1^\varepsilon \nabla^\perp v_2^\varepsilon) dx =: \sum_{i=1}^{5} T_i \qquad (4.54)
\end{aligned}$$

Now we estimate each term in the decomposition (4.54). First, applying the integration by parts and gradient's properties, we have that

$$
\begin{aligned}
T_1 &= -\mu_1 \int_{\mathbb{R}^2} \nabla\left((u_1^\varepsilon - K)_+^{p-1}\right)\cdot \nabla u_1^\varepsilon dx \\
&= -\mu_1 \int_{\mathbb{R}^2} \nabla\left((u_1^\varepsilon - K)_+^{p-1}\right)\cdot \nabla (u_1^\varepsilon - K) dx \\
&= -\frac{4(p-1)\mu_1}{p^2}\int_{\mathbb{R}^2}\left|\nabla\left((u_1^\varepsilon - K)_+^{p/2}\right)\right|^2 dx. \qquad (4.55)
\end{aligned}
$$

Second, the identity $\nabla\cdot\nabla^\perp v_i^\varepsilon = 0, i = 1,2$ gradient's properties and integration by parts yield the vanishing of $T_3 + T_5$, i.e.,

$$
\begin{aligned}
& T_3 + T_5 \\
&= -\chi_{11}\sin\alpha_{11}\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^{p-1}\left[\nabla u_1^\varepsilon\cdot\nabla^\perp v_1^\varepsilon + u_1^\varepsilon \nabla\cdot\nabla^\perp v_1^\varepsilon\right] dx \\
&- \chi_{12}\sin\alpha_{12}\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^{p-1}\left[\nabla u_1^\varepsilon\cdot\nabla^\perp v_2^\varepsilon + u_1^\varepsilon \nabla\cdot\nabla^\perp v_2^\varepsilon\right] dx \\
&= -\frac{\chi_{11}\sin\alpha_{11}}{p}\int_{\mathbb{R}^2}\nabla(u_1^\varepsilon - K)_+^p\cdot\nabla^\perp v_1^\varepsilon dx \\
&- \frac{\chi_{12}\sin\alpha_{12}}{p}\int_{\mathbb{R}^2}\nabla(u_1^\varepsilon - K)_+^p\cdot\nabla^\perp v_2^\varepsilon dx \\
&= \frac{\chi_{11}\sin\alpha_{11}}{p}\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^p\nabla\cdot\nabla^\perp v_1^\varepsilon dx \\
&+ \frac{\chi_{12}\sin\alpha_{12}}{p}\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^p\nabla\cdot\nabla^\perp v_2^\varepsilon dx \\
&= 0. \qquad (4.56)
\end{aligned}
$$

Now we estimate the second term T_2 as follows:

$$
\begin{aligned}
T_2 + T_4 &= -\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^{p-1}\nabla\cdot(u_1^\varepsilon\nabla\left(\delta_{11}w_1^\varepsilon + \delta_{12}w_2^\varepsilon\right))dx \\
&= -\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^{p-1}\nabla u_1^\varepsilon\cdot\nabla\left(\delta_{11}w_1^\varepsilon + \delta_{12}w_2^\varepsilon\right) dx \\
&- \int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^{p-1}u_1^\varepsilon \triangle\left(\delta_{11}w_1^\varepsilon + \delta_{12}w_2^\varepsilon\right) dx \\
&= -\frac{1}{p}\int_{\mathbb{R}^2}\nabla(u_1^\varepsilon - K)_+^p\cdot\nabla\left(\delta_{11}w_1^\varepsilon + \delta_{12}w_2^\varepsilon\right) dx \\
&- \int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^{p-1}u_1^\varepsilon \triangle\left(\delta_{11}w_1^\varepsilon + \delta_{12}w_2^\varepsilon\right) dx \\
&= -\frac{1}{p}\int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^p\left(-\triangle\left(\delta_{11}w_1^\varepsilon + \delta_{12}w_2^\varepsilon\right)\right) dx \\
&+ \int_{\mathbb{R}^2}(u_1^\varepsilon - K)_+^{p-1}u_1^\varepsilon\left(-\triangle\left(\delta_{11}w_1^\varepsilon + \delta_{12}w_2^\varepsilon\right)\right) dx.
\end{aligned}
$$

Next, we use $-\triangle(\delta_{11}w_1^\varepsilon+\delta_{12}w_2^\varepsilon)=-\triangle\mathbf{K}^\varepsilon*(\delta_{11}u_1^\varepsilon+\delta_{12}u_2^\varepsilon)$ to obtain

$$\begin{aligned}T_2+T_4&=-\frac{1}{p}\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^p\left(-\triangle\mathbf{K}^\varepsilon*(\delta_{11}u_1^\varepsilon+\delta_{12}u_2^\varepsilon)\right)dx\\&+\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p-1}u_1^\varepsilon\left(-\triangle\mathbf{K}^\varepsilon*(\delta_{11}u_1^\varepsilon+\delta_{12}u_2^\varepsilon)\right)dx\\&=\frac{p-1}{p}\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^p\left(-\triangle\mathbf{K}^\varepsilon*(\delta_{11}u_1^\varepsilon+\delta_{12}u_2^\varepsilon)\right)dx\\&+K\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p-1}\left(-\triangle\mathbf{K}^\varepsilon*(\delta_{11}u_1^\varepsilon+\delta_{12}u_2^\varepsilon)\right)dx.\end{aligned}$$

Using the fact that $(-\triangle\mathbf{K}^\varepsilon*K)=K\left\|-\triangle\mathbf{K}^\varepsilon\right\|_{L^1}=K$, we have

$$\begin{aligned}&T_2+T_4\\&=\frac{p-1}{p}\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^p\left(-\triangle\mathbf{K}^\varepsilon*(\delta_{11}(u_1^\varepsilon-K)+\delta_{12}(u_2^\varepsilon-K))\right)dx\\&+\frac{(p-1)(\delta_{11}+\delta_{12})}{p}K\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^p dx\\&+K\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p-1}\left(-\triangle\mathbf{K}^\varepsilon*(\delta_{11}(u_1^\varepsilon-K)+\delta_{12}(u_2^\varepsilon-K))\right)dx\\&+(\delta_{11}+\delta_{12})K^2\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p-1}dx.\end{aligned}$$

Then

$$\begin{aligned}&T_2+T_4\\&\le\frac{p-1}{p}\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^p\left(-\triangle\mathbf{K}^\varepsilon*\left(\delta_{11}(u_1^\varepsilon-K)_++\delta_{12}(u_2^\varepsilon-K)_+\right)\right)dx\\&+\frac{(p-1)(\delta_{11}+\delta_{12})}{p}K\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^p dx\\&+K\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p-1}\left(-\triangle\mathbf{K}^\varepsilon*\left(\delta_{11}(u_1^\varepsilon-K)_++\delta_{12}(u_2^\varepsilon-K)_+\right)\right)dx\\&+(\delta_{11}+\delta_{12})K^2\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p-1}dx.\end{aligned}$$

By Young's convolution inequality (3.91), we get

$$\begin{aligned}&\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^p\left(-\triangle\mathbf{K}^\varepsilon*(u_1^\varepsilon-K)_+\right)dx\\&\le\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p+1}dx.\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^p\left(-\triangle\mathbf{K}^\varepsilon*(u_2^\varepsilon-K)_+\right)dx\\&\le\left(\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^{p+1}dx\right)^{\frac{1}{p+1}}\left(\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p+1}dx\right)^{\frac{p}{p+1}}.\end{aligned}$$

By Young's inequality for products we have that

$$\begin{aligned}&\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^p\left(-\triangle\mathbf{K}^\varepsilon*(u_2^\varepsilon-K)_+\right)dx\\&\le\frac{p}{p+1}\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p+1}dx+\frac{1}{p+1}\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^{p+1}\,dx.\end{aligned}$$

So, we obtain that

$$\begin{aligned}&T_2+T_4\\&\le\frac{(p-1)\left[\delta_{11}(p+1)+\delta_{12}p\right]}{p(p+1)}\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p+1}dx\\&+\frac{(2p-1)\,\delta_{11}+2\,(p-1)\,\delta_{12}}{p}K\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p}dx\\&+(\delta_{11}+\delta_{12})\,K^2\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p-1}dx\\&+\frac{(p-1)\delta_{12}}{p(p+1)}\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^{p+1}dx+\frac{\delta_{12}}{p}K\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^{p}dx.\end{aligned}\tag{4.57}$$

Substituting (4.55), (4.56) and (4.57) into (4.54), we get

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^p\,dx\le-\frac{4(p-1)\mu_1}{p^2}\int_{\mathbb{R}^2}\left|\nabla\left((u_1^\varepsilon-K)_+^{p/2}\right)\right|^2dx\\&+\frac{(p-1)\left[\delta_{11}(p+1)+\delta_{12}p\right]}{p(p+1)}\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p+1}dx\\&+\frac{(2p-1)\,\delta_{11}+2\,(p-1)\,\delta_{12}}{p}K\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p}dx\\&+(\delta_{11}+\delta_{12})\,K^2\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p-1}dx\\&+\frac{(p-1)\delta_{12}}{p(p+1)}\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^{p+1}dx+\frac{\delta_{12}}{p}K\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^{p}dx.\end{aligned}\tag{4.58}$$

Similarly

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^p\,dx\le-\frac{4(p-1)\mu_2}{p^2}\int_{\mathbb{R}^2}\left|\nabla\left((u_2^\varepsilon-K)_+^{p/2}\right)\right|^2dx\\&+\frac{(p-1)\left[\delta_{22}(p+1)+\delta_{21}p\right]}{p(p+1)}\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^{p+1}dx\\&+\frac{(2p-1)\,\delta_{22}+2\,(p-1)\,\delta_{21}}{p}K\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^{p}dx\\&+(\delta_{22}+\delta_{21})\,K^2\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^{p-1}dx\\&+\frac{(p-1)\delta_{21}}{p(p+1)}\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p+1}dx+\frac{\delta_{21}}{p}K\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p}dx.\end{aligned}\tag{4.59}$$

The expression p(4.58)+p(4.59) gives

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}\left((u_1^\varepsilon-K)_+^p+(u_2^\varepsilon-K)_+^p\right)dx\\
&\leq -\frac{4(p-1)}{p}\mu_1\int_{\mathbb{R}^2}\left|\nabla\left((u_1^\varepsilon-K)_+^{p/2}\right)\right|^2dx\\
&+\frac{(p-1)\left[\delta_{11}(p+1)+\delta_{12}p+\delta_{21}\right]}{(p+1)}\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p+1}dx\\
&-\frac{4(p-1)}{p}\mu_2\int_{\mathbb{R}^2}\left|\nabla\left((u_2^\varepsilon-K)_+^{p/2}\right)\right|^2dx\\
&+\frac{(p-1)\left[\delta_{22}(p+1)+\delta_{21}p+\delta_{12}\right]}{(p+1)}\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^{p+1}dx\\
&+((2p-1)\,\delta_{11}+2\,(p-1)\,\delta_{12}+\delta_{21})K\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^pdx\\
&+((2p-1)\,\delta_{22}+2\,(p-1)\,\delta_{21}+\delta_{12})K\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^pdx\\
&+p\,(\delta_{11}+\delta_{12})\,K^2\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^{p-1}dx\\
&+p\,(\delta_{22}+\delta_{21})\,K^2\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^{p-1}dx.
\end{aligned}$$

The term involving $\int_{\mathbb{R}^2}(u_i^\varepsilon-K)_+^{p-1}dx$ can be estimated as follows:

$$\begin{aligned}
\int_{\mathbb{R}^2}(u_i^\varepsilon-K)_+^{p-1}dx &\leq \int_{K<u_i^\varepsilon\leq K+1}1dx+\int_{u_i^\varepsilon>K+1}(u_i^\varepsilon-K)_+^pdx\\
&\leq \frac{1}{K}\int_{K<u_i^\varepsilon\leq K+1}u_i^\varepsilon dx+\int_{u_i^\varepsilon>K+1}(u_i^\varepsilon-K)_+^pdx\\
&\leq \frac{\theta_i}{K}+\int_{\mathbb{R}^2}(u_i^\varepsilon-K)_+^pdx.
\end{aligned}$$

Applying Gagliardo-Nirenberg-Sobolev inequality (3.95), we got

$$\begin{aligned}
\int_{\mathbb{R}^2}(u_i^\varepsilon-K)_+^{p+1}dx &\leq C_{GNS}^2\left(\int_{\mathbb{R}^2}\left|\nabla\,(u_i^\varepsilon-K)_+^{\frac{p+1}{2}}\right|dx\right)^2\\
&= K_p\left(\int_{\mathbb{R}^2}\left|(u_i^\varepsilon-K)_+^{1/2}\,\nabla\left((u_i^\varepsilon-K)_+^{p/2}\right)\right|dx\right)^2\\
&\leq K_pM_i(K)\int_{\mathbb{R}^2}\left|\nabla\left((u_i^\varepsilon-K)_+^{p/2}\right)\right|^2dx.
\end{aligned}$$

where $K_p := C_{GNS}^2\left(1+\frac{1}{p}\right)^2$. So, we have that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}\left((u_1^\varepsilon-K)_+^p+(u_2^\varepsilon-K)_+^p\right)dx\\
&\leq\left(-\tfrac{4(p-1)}{p}\mu_1+\tfrac{K_pM_1(K)(p-1)[\delta_{11}(p+1)+\delta_{12}p+\delta_{21}]}{(p+1)}\right)\int_{\mathbb{R}^2}\left|\nabla\left((u_1^\varepsilon-K)_+^{p/2}\right)\right|^2dx\\
&+\left(-\tfrac{4(p-1)}{p}\mu_2+\tfrac{K_pM_2(K)(p-1)[\delta_{22}(p+1)+\delta_{21}p+\delta_{12}]}{(p+1)}\right)\int_{\mathbb{R}^2}\left|\nabla\left((u_2^\varepsilon-K)_+^{p/2}\right)\right|^2dx\\
&+\left((2p-1+pK)\,\delta_{11}+(pK+2p-2)\,\delta_{12}+\delta_{21}\right)K\int_{\mathbb{R}^2}(u_1^\varepsilon-K)_+^pdx\\
&+\left((2p-1+pK)\,\delta_{22}+(pK+2p-2)\,\delta_{21}+\delta_{12}\right)K\int_{\mathbb{R}^2}(u_2^\varepsilon-K)_+^pdx\\
&+pK\left((\delta_{11}+\delta_{12})\,\theta_1+(\delta_{22}+\delta_{21})\,\theta_2\right).
\end{aligned}$$

By choosing K sufficiently large such that

$$M_1(K)<\frac{4(p+1)\mu_1}{pK_p\left[\delta_{11}(p+1)+\delta_{12}p+\delta_{21}\right]},$$

and

$$M_2(K)<\frac{4(p+1)\mu_2}{pK_p\left[\delta_{22}(p+1)+\delta_{21}p+\delta_{12}\right]}.$$

Then, for a fixed interval $[0,T]$ with T arbitrarily large

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}\left((u_1^\varepsilon-K)_+^p+(u_2^\varepsilon-K)_+^p\right)dx\\
&\leq C_1\int_{\mathbb{R}^2}\left((u_1^\varepsilon-K)_+^p+(u_2^\varepsilon-K)_+^p\right)dx+C_2,
\end{aligned}$$

with

$$C_1:=(2p-1+pK)\left(\delta_{11}+\delta_{22}+\delta_{12}+\delta_{21}\right)K,$$

and

$$C_2:=pK\left((\delta_{11}+\delta_{12})\,\theta_1+(\delta_{22}+\delta_{21})\,\theta_2\right).$$

By Gronwall's inequality (differential form) [39, p. 624], we have that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left((u_1^\varepsilon-K)_+^p+(u_2^\varepsilon-K)_+^p\right)(x,t)dx\\
&\leq e^{C_1T}\left(\int_{\mathbb{R}^2}\left((u_{10}-K)_+^p+(u_{20}-K)_+^p\right)dx+C_2T\right).
\end{aligned}$$

So, we have that $\int_{\mathbb{R}^2}(u_i^\varepsilon-K)_+^p\,dx, i=1,2$ is finite on $[0,T]$ .Therefore, for any $t\in[0,T]$

$$\begin{aligned}
&\|u_i^\varepsilon(t)\|_{L^\infty([0,T];L^p(\mathbb{R}^2))}\\
&\leq\left\|(u_i^\varepsilon-K)_+\right\|_{L^\infty([0,T];L^p(\mathbb{R}^2))}+\|\min\{u_i^\varepsilon,K\}\|_{L^\infty([0,T];L^p(\mathbb{R}^2))}\\
&\leq e^{\frac{C_1}{p}T}\left(\int_{\mathbb{R}^2}\left((u_{10}-K)_+^p+(u_{20}-K)_+^p\right)dx+C_2T\right)^{\frac{1}{p}}+K^{\frac{p-1}{p}}\theta_i^{\frac{1}{p}},\qquad(4.60)
\end{aligned}$$

for any $p\in(1,\infty)$. ■

Extra Uniform estimates.

Lemma 41 *Assume that* $0 \leq u_{10}, u_{20} \in L^1(\mathbb{R}^2, \ln(1+|x|^2)dx) \cap L^\infty(\mathbb{R}^2)$, $u_{10}\ln u_{10},\ u_{20}\ln u_{20} \in L^1(\mathbb{R}^2, dx)$ *and* (θ_1, θ_2) *satisfies*

$$\delta_{11}\theta_1 < 8\pi\mu_1,\ \ \delta_{22}\theta_2 < 8\pi\mu_2,$$
$$\text{and } \frac{8\pi\mu_1}{\delta_{12}}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right) > 0.$$

Consider a non-negative solution of (4.21) such that $u_1^\varepsilon, u_2^\varepsilon$ *are bounded in* L^∞_{loc} $(\mathbb{R}^+, L^p(\mathbb{R}^2)), 1 < p \leq \infty$. *Then, with bounds independent on* ε, *we have for all* $T > 0$:

(i) *The function* $(t,x) \longmapsto \left|\nabla (u_i^\varepsilon)^{p/2}\right|(x,t)$ *is bounded in* $L^2([0,T]\,; L^2(\mathbb{R}^2))$, *for any* $1 \leq p < \infty$.

(ii) *The function* $(t,x) \longmapsto |\nabla v_i^\varepsilon|(x,t), i = 1,2,$ *is bounded in* $L^\infty([0,T]\,; L^p(\mathbb{R}^2))$, *for any* $2 < p \leq \infty$.

(iii) *The function* $(t,x) \longmapsto \left|u_i^\varepsilon A_{ij} \nabla v_j^\varepsilon\right|(x,t), i,j = 1,2,$ *is bounded in* $L^2([0,T];\ L^2(\mathbb{R}^2))$.

(iv) *The function* $(t,x) \longmapsto u_i^\varepsilon(x,t)\ln(1+|x|^2), i = 1,2,$ *is bounded in* $L^\infty([0,T], L^1(\mathbb{R}^2))$.

(v) The function $(t,x) \longmapsto u_i^\varepsilon(x,t)\ln u_i^\varepsilon(x,t), i = 1,2,$ is bounded in $L^\infty([0,T],\ L^1(\mathbb{R}^2))$.

(vi) *The function* $(t,x) \longmapsto \partial_t u_i^\varepsilon(x,t), i = 1,2,$ *is bounded in* $L^2([0,T], H^1(\mathbb{R}^2)^*)$.

(vii) *The function* $(t,x) \longmapsto \sqrt{u_i^\varepsilon}\left|\nabla v_j^\varepsilon\right|(x,t), i,j = 1,2,$ *is bounded in* $L^2([0,T]\ ; L^2(\mathbb{R}^2))$.

Proof. The proof follows the same argument of Lemma 22 with minor modifications. ■

Strong convergence of u_i^ε. To demonstrate the strong convergence of u_i^ε in $L^2([0,T]\,; L^2(\mathbb{R}^2))$, we will once again employ the Aubin-Lions compactness method.

Considering the embeddings

$$H^1(\Omega) \overset{\text{Compact}}{\hookrightarrow} L^2(\Omega) \overset{\text{Continuous}}{\hookrightarrow} H^1(\Omega)^*,$$

where Ω is a bounded open set of class C^1. By Lemma 23, we have that for any Ω there exists a subsequence,still denoted by $u_i^\varepsilon, i = 1,2$, such that

$$u_i^\varepsilon \to u_i \text{ in } L^2([0,T]\,; L^2(\Omega)).$$

By a diagonal argument, the following uniform strong convergence holds true that for any $R > 0$

$$u_i^\varepsilon \to u_i \text{ in } L^2([0,T]\,; L^2(B_R(0))). \tag{4.61}$$

Now, to extend (4.61) to the whole space, we observe that

$$\begin{aligned}&\int_0^T \|u_i^\varepsilon\|^2_{L^2(|x|>R)}\,dt\\&=\int_0^T\int_{|x|>R}(u_i^\varepsilon)^2\,dxdt\le\frac{1}{\sqrt{\ln(1+R^2)}}\int_0^T\int_{\mathbb{R}^2}\left(\ln(1+|x|^2)\right)^{1/2}(u_i^\varepsilon)^2\,dxdt\\&\le\frac{1}{\sqrt{\ln(1+R^2)}}\int_0^T\left(\|u_i^\varepsilon\|^{3/2}_{L^3(\mathbb{R}^2)}\left(\int_{\mathbb{R}^2}\ln(1+|x|^2)u_i^\varepsilon dx\right)^{1/2}\right)dt\to 0,\end{aligned}$$

as $R\to\infty$ and the weak semi-continuity of $L^2([0,T]\,;L^2(\mathbb{R}^2))$ implies

$$\int_0^T\|u_i\|^2_{L^2(|x|>R)}\,dt\le\liminf_{\varepsilon\to 0}\int_0^T\|u_i^\varepsilon\|^2_{L^2(|x|>R)}\,dt\to 0\text{ as }R\to\infty.$$

Therefore,

$$\begin{aligned}&\int_0^T\|u_i^\varepsilon-u_i\|^2_{L^2(\mathbb{R}^2)}\,dt\\&\le 2\int_0^T\left(\|u_i^\varepsilon\|^2_{L^2(|x|>R)}+\|u_i\|^2_{L^2(|x|>R)}+\|u_i^\varepsilon-u_i\|^2_{L^2(|x|\le R)}\right)dt\to 0,\end{aligned}$$

as $R\to\infty,\varepsilon\to 0$. So, we have that

$$u_i^\varepsilon\to u_i\text{ in }L^2([0,T]\,;L^2(\mathbb{R}^2)).\tag{4.62}$$

By Proposition 24, there is subsequence, still denoted by u_i^ε, such that

$$u_i^\varepsilon(t)\to u_i(t)\text{ in }L^2(\mathbb{R}^2)\text{ for a.e. on }[0,T]\,.\tag{4.63}$$

Mass conservation. Multiplying the first equation of system (4.21) by any test function $\varphi(x)\in C_0^\infty(\mathbb{R}^2)$ and integrating over $[0,t)\times\mathbb{R}^2$

$$\begin{aligned}&\int_{\mathbb{R}^2}\varphi u_1^\varepsilon(x,t)dx-\int_{\mathbb{R}^2}\varphi u_{10}(x)dx\\&=\mu_1\int_0^t\int_{\mathbb{R}^2}u_1^\varepsilon\Delta\varphi dxd\tau+\chi_{11}\int_0^t\int_{\mathbb{R}^2}\nabla\varphi\cdot(u_1^\varepsilon A_{11}\nabla v_1^\varepsilon)dxd\tau\\&+\chi_{12}\int_0^t\int_{\mathbb{R}^2}\nabla\varphi\cdot(u_1^\varepsilon A_{12}\nabla v_2^\varepsilon)dxd\tau\end{aligned}$$

Letting $\varphi(x)=\varphi_R(x)$ be defined as in (3.72), we have

$$\left|\int_0^t\int_{\mathbb{R}^2}u_1^\varepsilon\Delta\varphi_R dxd\tau\right|\le\frac{C^*}{R^2}\theta_1 T,$$

and

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}^2}\nabla\varphi\cdot(u_1^\varepsilon A_{ij}\nabla v_j^\varepsilon)dxd\tau\right|&\le\frac{C^*}{R}\left\|\nabla v_j^\varepsilon\right\|_{L^\infty([0,T];L^\infty(\mathbb{R}^2))}\theta_1 T\\&\le\frac{C_3}{R}\theta_1 T, j=1,2.\end{aligned}$$

Due to (4.63),passing to the limit $\varepsilon \to 0, R \to \infty$, we obtain the mass conservation property

$$\int_{\mathbb{R}^2} u_1(x,t)dx = \theta_1.$$

Similarly,

$$\int_{\mathbb{R}^2} u_2(x,t)dx = \theta_2.$$

Existence of the weak solution. Now multiplying the first equation of system (4.21) by any test function $\varphi \in C_0^\infty(\mathbb{R}^2)$ and integrating over $[0,t)\times R^2$, we get the weak formulation for u_1^ε

$$\begin{aligned}
&\int_{\mathbb{R}^2} \varphi u_1^\varepsilon(x,t)dx - \int_{\mathbb{R}^2} \varphi u_{10}(x)dx \\
&= \mu_1 \int_0^t \int_{\mathbb{R}^2} u_1^\varepsilon \Delta\varphi dx d\tau + \chi_{11} \int_0^t \int_{\mathbb{R}^2} \nabla\varphi \cdot (u_1^\varepsilon A_{11} \nabla v_1^\varepsilon)\, dx d\tau \\
&+ \chi_{12} \int_0^t \int_{\mathbb{R}^2} \nabla\varphi \cdot (u_1^\varepsilon A_{12} \nabla v_2^\varepsilon)\, dx d\tau.
\end{aligned} \tag{4.64}$$

By following the same strategy as in Chapter 3, we can take the limit as $\varepsilon \to 0$ in equation (4.64) to establish the existence of a global weak solution for the system (4.8)-(4.10).

Boundedness of the second moment of the weak solution.

Lemma 42 *If* $u_{10}, u_{20} \in L^1(\mathbb{R}^2, |x|^2\, dx)$, *then* $|x|^2 u_i^\varepsilon, |x|^2 u_i \in L^\infty_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$ *for* $i = 1, 2$.

Proof. The proof follows the same argument of Lemma 25 with minor modifications. ■

The energy inequality of the weak solution. Integrating (4.23) in time from 0 to t follows

$$\begin{aligned}
&\frac{\mu_1}{\delta_{12}} \int_{\mathbb{R}^2} u_1^\varepsilon \ln u_1^\varepsilon dx + \frac{\mu_2}{\delta_{21}} \int_{\mathbb{R}^2} u_2^\varepsilon \ln u_2^\varepsilon dx \\
&- \frac{\chi_{11}\cos\alpha_{11}}{2\delta_{12}} \int_{\mathbb{R}^2} u_1^\varepsilon v_1^\varepsilon dx - \frac{\chi_{12}\cos\alpha_{12}}{2\delta_{12}} \int_{\mathbb{R}^2} u_1^\varepsilon v_2^\varepsilon dx \\
&- \frac{\chi_{21}\cos\alpha_{21}}{2\delta_{21}} \int_{\mathbb{R}^2} u_2^\varepsilon v_1^\varepsilon dx - \frac{\chi_{22}\cos\alpha_{22}}{2\delta_{21}} \int_{\mathbb{R}^2} u_2^\varepsilon v_2^\varepsilon dx \\
&+ \frac{1}{\delta_{12}} \int_0^t \int_{\mathbb{R}^2} u_1^\varepsilon \left|\nabla(\mu_1 \ln u_1^\varepsilon - \chi_{11}\cos\alpha_{11} v_1^\varepsilon - \chi_{12}\cos\alpha_{12} v_2^\varepsilon)\right|^2 dx dt \\
&+ \frac{1}{\delta_{21}} \int_0^t \int_{\mathbb{R}^2} u_2^\varepsilon \left|\nabla(\mu_2 \ln u_2^\varepsilon - \chi_{21}\cos\alpha_{21} v_1^\varepsilon - \chi_{22}\cos\alpha_{22} v_2^\varepsilon)\right|^2 dx dt \\
&= E(0)
\end{aligned} \tag{4.65}$$

By following the same strategy as in Chapter 3, we can take the limit as $\varepsilon \to 0$ in equation (4.65) to derive the energy dissipation (4.16).

4.1.2 Proof of theorem 34

Let $p > 1$, multiplying the first equation of system (4.21) by $(u_1^\varepsilon)^{p-1}$ and integrating over $\mathbb{R}^2$, we get

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2} |u_1^\varepsilon|^p \, dx = \mu_1 \int_{\mathbb{R}^2} (u_1^\varepsilon)^{p-1} \, \triangle u_1^\varepsilon dx \\
&- \chi_{11} \int_{\mathbb{R}^2} (u_1^\varepsilon)^{p-1} \, \nabla \cdot (u_1^\varepsilon A_{11} \nabla v_1^\varepsilon) dx \\
&- \chi_{12} \int_{\mathbb{R}^2} (u_1^\varepsilon)^{p-1} \, \nabla \cdot (u_1^\varepsilon A_{12} \nabla v_2^\varepsilon) dx \\
&= \mu_1 \int_{\mathbb{R}^2} (u_1^\varepsilon)^{p-1} \, \triangle u_1^\varepsilon dx - \chi_{11} \cos\alpha_{11} \int_{\mathbb{R}^2} (u_1^\varepsilon)^{p-1} \, \nabla \cdot (u_1^\varepsilon \nabla v_1^\varepsilon) dx \\
&- \chi_{11} \sin\alpha_{11} \int_{\mathbb{R}^2} (u_1^\varepsilon)^{p-1} \, \nabla \cdot (u_1^\varepsilon \nabla^\perp v_1^\varepsilon) dx \\
&- \chi_{12} \cos\alpha_{12} \int_{\mathbb{R}^2} (u_1^\varepsilon)^{p-1} \, \nabla \cdot (u_1^\varepsilon \nabla v_2^\varepsilon) dx \\
&- \chi_{12} \sin\alpha_{12} \int_{\mathbb{R}^2} (u_1^\varepsilon)^{p-1} \, \nabla \cdot (u_1^\varepsilon \nabla^\perp v_2^\varepsilon) dx \\
&=: \sum_{i=1}^{5} T_i.
\end{aligned}$$

Now we estimate T_1 applying the integration by parts and gradient's properties as follows

$$\begin{aligned}
T_1 &= -\mu_1 \int_{\mathbb{R}^2} \nabla \, (u_1^\varepsilon)^{p-1} \cdot \nabla u_1^\varepsilon dx \\
&= -\mu_1 \int_{\mathbb{R}^2} \nabla \left((u_1^\varepsilon)^{p/2} \right)^{\frac{2(p-1)}{p}} \cdot \nabla \left((u_1^\varepsilon)^{p/2} \right)^{2/p} dx \\
&= -\frac{4(p-1)\mu_1}{p^2} \int_{\mathbb{R}^2} (u_1^\varepsilon)^{p/2-1} \, \nabla \, (u_1^\varepsilon)^{p/2} \cdot (u_1^\varepsilon)^{1-p/2} \, \nabla \, (u_1^\varepsilon)^{p/2} \, dx \\
&= -\frac{4(p-1)}{p^2} \mu_1 \int_{\mathbb{R}^2} \left| \nabla \, (u_1^\varepsilon)^{p/2} \right|^2 dx.
\end{aligned}$$

By the fact that $\nabla \cdot \nabla^\perp v^\varepsilon = 0$, we have that $T_3 + T_5 = 0$. Indeed,

$$\begin{aligned}
&T_3 + T_5 \\
&= -\chi_{11} \sin\alpha_{11} \int_{\mathbb{R}^2} (u_1^\varepsilon)^{p-1} \left[\nabla u_1^\varepsilon \cdot \nabla^\perp v_1^\varepsilon + u_1^\varepsilon \nabla \cdot \nabla^\perp v_1^\varepsilon \right] dx \\
&- \chi_{12} \sin\alpha_{12} \int_{\mathbb{R}^2} (u_1^\varepsilon)^{p-1} \left[\nabla u_1^\varepsilon \cdot \nabla^\perp v_2^\varepsilon + u_1^\varepsilon \nabla \cdot \nabla^\perp v_2^\varepsilon \right] dx \\
&= -\frac{\chi_{11} \sin\alpha_{11}}{p} \int_{\mathbb{R}^2} \nabla \, (u_1^\varepsilon)^p \cdot \nabla^\perp v_1^\varepsilon dx
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\chi_{12}\sin\alpha_{12}}{p}\int_{\mathbb{R}^2}\nabla\left(u_1^\varepsilon\right)^p\cdot\nabla^\perp v_2^\varepsilon dx\\
&= \frac{\chi_{11}\sin\alpha_{11}}{p}\int_{\mathbb{R}^2}\left(u_1^\varepsilon\right)^p\nabla\cdot\nabla^\perp v_1^\varepsilon dx\\
&+ \frac{\chi_{12}\sin\alpha_{12}}{p}\int_{\mathbb{R}^2}\left(u_1^\varepsilon\right)^p\nabla\cdot\nabla^\perp v_2^\varepsilon dx\\
&= 0.
\end{aligned}
$$

Next we estimate T_2+T_4 as follows:

$$
\begin{aligned}
T_2+T_4 &= -\int_{\mathbb{R}^2}\left(u_1^\varepsilon\right)^{p-1}\nabla\cdot\left(u_1^\varepsilon\nabla\left(\delta_{11}w_1^\varepsilon+\delta_{12}w_2^\varepsilon\right)\right)dx\\
&= -\int_{\mathbb{R}^2}\left(u_1^\varepsilon\right)^{p-1}\nabla u_1^\varepsilon\cdot\nabla\left(\delta_{11}w_1^\varepsilon+\delta_{12}w_2^\varepsilon\right)dx\\
&- \int_{\mathbb{R}^2}\left(u_1^\varepsilon\right)^{p-1}u_1^\varepsilon\triangle\left(\delta_{11}w_1^\varepsilon+\delta_{12}w_2^\varepsilon\right)dx\\
&= -\frac{1}{p}\int_{\mathbb{R}^2}\nabla\left(u_1^\varepsilon\right)^p\cdot\nabla\left(\delta_{11}w_1^\varepsilon+\delta_{12}w_2^\varepsilon\right)dx\\
&- \int_{\mathbb{R}^2}\left(u_1^\varepsilon\right)^p\triangle\left(\delta_{11}w_1^\varepsilon+\delta_{12}w_2^\varepsilon\right)dx\\
&= \frac{p-1}{p}\int_{\mathbb{R}^2}\left(u_1^\varepsilon\right)^p\left(-\triangle\left(\delta_{11}w_1^\varepsilon+\delta_{12}w_2^\varepsilon\right)\right)dx.
\end{aligned}
$$

Therefore, we arrive at the following estimate

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}\left|u_1^\varepsilon\right|^p dx\\
&= -\frac{4(p-1)}{p}\mu_1\int_{\mathbb{R}^2}\left|\nabla\left(u_1^\varepsilon\right)^{p/2}\right|^2 dx\\
&+ (p-1)\int_{\mathbb{R}^2}\left(u_1^\varepsilon\right)^p\left(-\triangle\left(\delta_{11}w_1^\varepsilon+\delta_{12}w_2^\varepsilon\right)\right)dx\\
&= -\tfrac{4(p-1)}{p}\mu_1\int_{\mathbb{R}^2}\left|\nabla\left(u_1^\varepsilon\right)^{p/2}\right|^2 dx\\
&+ (p-1)\delta_{11}\int_{\mathbb{R}^2}\underbrace{\left(u_1^\varepsilon\right)^p\left(-\triangle\mathbf{K}^\varepsilon*\left(u_1^\varepsilon\right)\right)}_{>0}dx\\
&+ (p-1)\delta_{12}\int_{\mathbb{R}^2}\underbrace{\left(u_1^\varepsilon\right)^p\left(-\triangle\mathbf{K}^\varepsilon*\left(u_2^\varepsilon\right)\right)}_{>0}dx.
\end{aligned}
$$

Since $\delta_{11},\delta_{12}\le 0$, we have that $\int_{\mathbb{R}^2}|u_1^\varepsilon|^p\,dx$ is non-increasing and thus bounded from above due to the assumption $u_{10}\in L^p(\mathbb{R}^2)$. Similarly, $\int_{\mathbb{R}^2}|u_2^\varepsilon|^p\,dx$ is also bounded from above. Therefore, we have that for any initial masses $\theta_i, i=1,2$, the solution $(u_1^\varepsilon,u_2^\varepsilon)$ of (4.21) is bounded in $L^\infty_{loc}(\mathbb{R}^+,L^p(\mathbb{R}^2))$, for all $1<p<\infty$. Hence, under assumption (4.11), we can pass to the limit $\varepsilon\to 0$ by applying the same argument described in the previous case, which allow us to conclude the global existence of weak solution for system (4.8)-(4.10).

4.2 Finite time blow-up for radially symmetric solutions

Our purpose in this section is to derive sharp conditions on the initial masses for having blow-up for system (4.8)-(4.10). We follows the ideas for the multispecies case with rotational flux terms introduced in Chapter 3. The key remark is the lemma 27, which enables the proof of the possibility of having blow-up for radially symmetric solutions, leaving the question for non-radial case open.

Proposition 43 (Local existence) *Given $u_{10}, u_{20} \in L^1(\mathbb{R}^2)$, then there exists a maximal time $T_{\max} > 0$ of existence of a positive classic solution (u_1, u_2) to the system (4.8)-(4.10). Moreover, the masses $\int_{\mathbb{R}^2} u_1(\cdot, t)dx$ and $\int_{\mathbb{R}^2} u_2(\cdot, t)dx$ remain constants in time.*

Proof. The proof follows the same argument of Proposition 11 with minor modifications. ■

Theorem 44 *Let us denote by θ_i with $i = 1, 2$ the total initial masses define by (4.12). Consider a weak solution (u_1, u_2) of system (4.8)-(4.10) and let $[0, T_{\max})$ be the corresponding maximal interval of existence. Assume that the initial data u_{10}, u_{20} satisfy (4.11), radially symmetric, $u_{10}, u_{20} \in L^1(\mathbb{R}^2, (1 + |x|^2)dx)$ and $\chi_{ij}, a_{ij}, \alpha_{ij}, i = 1, 2,$ satisfy (4.13). If θ_1 and θ_2 satisfy any of the inequalities*

$$\delta_{11}\theta_1 > 8\pi\mu_1, \ \text{or } \delta_{22}\theta_2 > 8\pi\mu_2, \tag{4.66}$$

or

$$\frac{8\pi\mu_1}{\delta_{12}}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right) < 0, \tag{4.67}$$

then $T_{\max} < +\infty$.

Proof. *Let us first assume that θ_1 and θ_2 satisfy (4.67). We re-write the equation for u_i in the form*

$$\begin{aligned}\partial_t u_i = {} & \mu_i \triangle u_i - \chi_{i1}\cos\alpha_{i1}\nabla\cdot(u_i\nabla v_1) - \chi_{i2}\cos\alpha_{i2}\nabla\cdot(u_i\nabla v_2) \\ & - \chi_{i1}\sin\alpha_{i1}(\nabla u_i\cdot\nabla^{\perp}v_1) - \chi_{i2}\sin\alpha_{i2}(\nabla u_i\cdot\nabla^{\perp}v_2).\end{aligned}$$

After Lemma 27, the last terms vanishes, thus

$$\partial_t u_i = \mu_i \triangle u_i - \chi_{i1}\cos\alpha_{i1}\nabla\cdot(u_i\nabla v_1) - \chi_{i2}\cos\alpha_{i2}\nabla\cdot(u_i\nabla v_2). \tag{4.68}$$

Multiplying (4.68) by $|x|^2$ and integrating, we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 u_i dx \\ &= \mu_i \int_{\mathbb{R}^2} |x|^2 \triangle u_i dx - \chi_{i1}\cos\alpha_{i1}\int_{\mathbb{R}^2} |x|^2 \nabla\cdot(u_i\nabla v_1)dx \\ &- \chi_{i2}\cos\alpha_{i2}\int_{\mathbb{R}^2} |x|^2\nabla\cdot(u_i\nabla v_2)dx \\ &= 4\mu_i\theta_i + 2\chi_{i1}\cos\alpha_{i1}\int_{\mathbb{R}^2} x\cdot(u_i\nabla v_1)dx \\ &+ 2\chi_{i2}\cos\alpha_{i2}\int_{\mathbb{R}^2} x\cdot(u_i\nabla v_2)dx.\end{aligned} \tag{4.69}$$

Using the representation of $\nabla v_i, i = 1, 2,$ we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 u_1 dx \\
&= 4\mu_1\theta_1 \\
&- \frac{\chi_{11}\cos\alpha_{11}}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{x\cdot(x-y)}{|x-y|^2} u_1(x,t)(a_{11}u_1(y,t) + a_{12}u_2(y,t))dydx \\
&- \frac{\chi_{12}\cos\alpha_{12}}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{x\cdot(x-y)}{|x-y|^2} u_1(x,t)(a_{21}u_1(y,t) + a_{22}u_2(y,t))dydx. \quad (4.70)
\end{aligned}$$

Similarly

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 u_2 dx \\
&= 4\mu_2\theta_2 \\
&- \frac{\chi_{21}\cos\alpha_{21}}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{x\cdot(x-y)}{|x-y|^2} u_2(x,t)(a_{11}u_1(y,t) + a_{12}u_2(y,t))dydx \\
&- \frac{\chi_{22}\cos\alpha_{22}}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{x\cdot(x-y)}{|x-y|^2} u_2(x,t)(a_{21}u_1(y,t) + a_{22}u_2(y,t))dydx. \quad (4.71)
\end{aligned}$$

The expression $\frac{2\pi}{\delta_{12}}$(4.70)+$\frac{2\pi}{\delta_{21}}$(4.71) gives

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{2\pi}{\delta_{12}}\int_{\mathbb{R}^2}|x|^2 u_1 dx + \frac{2\pi}{\delta_{21}}\int_{\mathbb{R}^2}|x|^2 u_2 dx\right) \\
&= \frac{8\pi\mu_1}{\delta_{12}}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}}\theta_2 \\
&- 2\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{x\cdot(x-y)}{|x-y|^2}\left(u_1(x,t)u_2(y,t) + \frac{\delta_{11}}{\delta_{12}}u_1(x,t)u_1(y,t)\right)dydx \\
&- 2\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{x\cdot(x-y)}{|x-y|^2}\left(u_2(x,t)u_1(y,t) + \frac{\delta_{22}}{\delta_{21}}u_2(x,t)u_2(y,t)\right)dydx
\end{aligned}$$

$$\begin{aligned}
&= \frac{8\pi\mu_1}{\delta_{12}}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}}\theta_2 \\
&- 2\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{x\cdot(x-y)}{|x-y|^2}\left(u_1(x,t)u_2(y,t) + u_2(x,t)u_1(y,t)\right)dydx \\
&- \frac{2\delta_{11}}{\delta_{12}}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{x\cdot(x-y)}{|x-y|^2} u_1(x,t)u_1(y,t)dydx \\
&- \frac{2\delta_{22}}{\delta_{21}}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{x\cdot(x-y)}{|x-y|^2} u_2(x,t)u_2(y,t)dydx
\end{aligned}$$

The symmetry in the variables x and y in the last integrals implies

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{2\pi}{\delta_{12}}\int_{\mathbb{R}^2}|x|^2 u_1 dx + \frac{2\pi}{\delta_{21}}\int_{\mathbb{R}^2}|x|^2 u_2 dx\right)\\
&= \frac{8\pi\mu_1}{\delta_{12}}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}}\theta_2\\
&- \int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{(x-y)\cdot(x-y)}{|x-y|^2}\left(u_1(x,t)u_2(y,t)+u_2(x,t)u_1(y,t)\right)dydx\\
&- \frac{\delta_{11}}{\delta_{12}}\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{(x-y)\cdot(x-y)}{|x-y|^2}u_1(x,t)u_1(y,t)dydx\\
&- \frac{\delta_{22}}{\delta_{21}}\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{(x-y)\cdot(x-y)}{|x-y|^2}u_2(x,t)u_2(y,t)dydx\\
&= \frac{8\pi\mu_1}{\delta_{12}}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right).
\end{aligned}$$

Let the second moment $m(t)$ with respect to the origin for the whole population, defined by

$$m(t) := \frac{2\pi}{\delta_{12}}\int_{\mathbb{R}^2}|x|^2 u_1 dx + \frac{2\pi}{\delta_{21}}\int_{\mathbb{R}^2}|x|^2 u_2 dx.$$

Thus

$$\frac{d}{dt}m(t) = \frac{8\pi\mu_1}{\delta_{12}}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right).$$

Integrating on $(0,t)$, we obtain that

$$m(t) = m(0) + \left(\frac{8\pi\mu_1}{\delta_{12}}\theta_1 + \frac{8\pi\mu_2}{\delta_{21}}\theta_2 - \left(\frac{\delta_{11}}{\delta_{12}}\theta_1^2 + 2\theta_1\theta_2 + \frac{\delta_{22}}{\delta_{21}}\theta_2^2\right)\right)t \tag{4.72}$$

The inequality (4.67) implies now that $m(t)$ should become negative in finite time which is impossible since u_1 and u_2 are non-negative and $\delta_{12}, \delta_{21} > 0$. In conclusion $T_{\max} < \infty$. We proceed now to show that each of the inequalities in (4.66) imply $T_{\max} < \infty$. In this case, we defined the second moment $m_i(t)$ with respect to the origin for each variable

$$m_i(t) := \int_{\mathbb{R}^2}|x|^2 u_i(x,t)dx.$$

as well as the cumulative mass $M_i(r,t)$

$$M_i(r,t) := \int_{B(0,r)} u_i(x,t)dx = 2\pi\int_0^r u_i(\rho,t)\rho d\rho.$$

By (4.69) we have that

$$\begin{aligned}
\frac{d}{dt}m_i(t) = 4\mu_i\theta_i + 2\chi_{i1}\cos\alpha_{i1}\int_{\mathbb{R}^2} x\cdot(u_i\nabla v_1)dx\\
+ 2\chi_{i2}\cos\alpha_{i2}\int_{\mathbb{R}^2} x\cdot(u_i\nabla v_2)dx.
\end{aligned} \tag{4.73}$$

In polar coordinates

$$-\triangle v_j = -\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_j}{dr}\right) = a_{j1}u_1 + a_{j2}u_2, j = 1,2.$$

Thus

$$\begin{aligned} r\frac{dv_j}{dr} &= -a_{j1}\int_0^r u_1(\rho,t)\rho d\rho - a_{j2}\int_0^r u_2(\rho,t)\rho d\rho \\ &= -\frac{a_{j1}}{2\pi}\int_{B(0,r)} u_1(x,t)dx - \frac{a_{j2}}{2\pi}\int_{B(0,r)} u_2(x,t)dx \\ &= -\frac{a_{j1}}{2\pi}M_1(r,t) - \frac{a_{j2}}{2\pi}M_2(r,t), j=1,2. \end{aligned} \tag{4.74}$$

Moreover $x\cdot\nabla f = r\frac{df}{dr}$, *then*

$$\begin{aligned} &\chi_{ij}\cos\alpha_{ij}\int_{\mathbb{R}^2} x\cdot(u_i\nabla v_j)dx \\ &= 2\pi\chi_{ij}\cos\alpha_{ij}\int_0^{+\infty} u_i\rho\frac{dv_j}{d\rho}\rho d\rho, i,j=1,2. \end{aligned} \tag{4.75}$$

Combining (4.74) and (4.75), we get

$$\begin{aligned} &\chi_{i1}\cos\alpha_{i1}\int_{\mathbb{R}^2} x\cdot(u_i\nabla v_1)dx + \chi_{i2}\cos\alpha_{i2}\int_{\mathbb{R}^2} x\cdot(u_i\nabla v_2)dx \\ &= -\delta_{11}\int_0^{+\infty} M_1(r,t)u_i(\rho,t)\rho d\rho - \delta_{22}\int_0^{+\infty} M_2(r,t)u_i(\rho,t)\rho d\rho \\ &\leq -\delta_{ii}\int_0^{+\infty} M_i(r,t)u_i(\rho,t)\rho d\rho = -\frac{\delta_{ii}}{2\pi}\int_0^{+\infty} M_i\frac{dM_i}{d\rho}d\rho \\ &= -\frac{\delta_{ii}}{4\pi}\int_0^{+\infty}\frac{d}{d\rho}M_i^2 d\rho = -\frac{\delta_{ii}}{4\pi}\theta_i^2, \end{aligned} \tag{4.76}$$

since $\delta_{11}, \delta_{22} \geq 0$. *Replacing (4.76) in (4.73), we obtain*

$$\frac{d}{dt}m_i(t) \leq 4\mu_i\theta_i - \frac{\delta_{ii}}{2\pi}\theta_i^2 = 4\mu_i\theta_i\left(1 - \frac{\delta_{ii}}{8\pi\mu_i}\theta_i\right), i=1,2. \tag{4.77}$$

Therefore, we have $T_{\max} < \infty$ *when*

$$\delta_{ii}\theta_i > 8\pi\mu_i, i=1,2.$$

For the sake of simplicity, we have just performed a formal proof. However, this argument can be made rigorous by taking in the weak formulation the test function $|x|^2\varphi_R(x) \in C_0^\infty(\mathbb{R}^2)$, *where* $\varphi_R(x)$ *is defined as in (3.72), which grows to* $|x|^2$ *as* $R\to\infty$. *Then, we can pass to the limit using Lemma 27 and the fact that* $\Delta(|x|^2\varphi_R(x))$ *remains bounded and* $\nabla(|x|^2\varphi_R(x))$ *is Lipschitz continuous.* ■

4.3 Discussion

We would like to summarize in this section the biological interpretation of the results given in sections two and three in the context of paracrine and autocrine signalling loops when cells are surrounded by a rotational flux. Thus, we recall that its dynamics is described by the mathematical model given by the equations (4.5)-(4.7). In this case the statement of Theorem 29 simplifies to state that

1. if $\alpha \in (-\pi/2, \pi/2)$ and (θ_1, θ_2) satisfies

$$\theta_1 < \frac{8\pi\mu_1}{a_{11}\chi_{11}\cos\alpha},$$

and

$$\frac{8\pi\mu_1}{a_{22}\chi_{12}\cos\alpha}\theta_1 + \frac{8\pi\mu_2}{a_{11}\chi_{21}\cos\alpha}\theta_2 - \left(\frac{a_{11}\chi_{11}}{a_{22}\chi_{12}}\theta_1^2 + 2\theta_1\theta_2\right) > 0,$$

or else,

2. if $\alpha \in (-\pi, -\pi/2] \cup [\pi/2, \pi]$ then $T_{\max} = \infty$.

We intepret this last result as conditions that guarantee that there is no cell aggregation. When compared with the results in reference [28]), we notice that this result show that the rotational flux not only can delay a blow-up but avoid it.

On the other hand, Theorem 29 item 2, implies that in case any of the inequalities

$$\theta_1 > \frac{8\pi\mu_1}{a_{11}\chi_{11}\cos\alpha}, \tag{4.78}$$

or

$$\frac{8\pi\mu_1}{a_{22}\chi_{12}\cos\alpha}\theta_1 + \frac{8\pi\mu_2}{a_{11}\chi_{21}\cos\alpha}\theta_2 - \left(\frac{a_{11}\chi_{11}}{a_{22}\chi_{12}}\theta_1^2 + 2\theta_1\theta_2\right) < 0, \tag{4.79}$$

the blow-up is possible when $\alpha \in (-\pi/2, \pi/2)$. We interpret this result as conditions predicting a possible CTCs clusters and risk of metastasis. In particular, the inequality (4.79) implies that even if

$$\theta_1 < \frac{8\pi\mu_1}{a_{11}\chi_{11}\cos\alpha}$$

still is possible to have blow-up. This results suggest also that macrophages can induce cell aggregation though the total mass of CTCs is small.

Chapter 5

Blow-up of solutions to the two-dimensional Keller-Segel model with tensorial flux

Abstract

In this chapter, we aim to demonstrate the possibility of having solutions blowing up in finite time when subjected to a tensorial flux of the form Av, where A represents an arbitrary 2×2 matrix with constant components satisfying $\mathrm{Tr}(A), \det(A) > 0$, through the design of a new technique. Unlike most current publications in the literature that impose conditions on $\|A\|$, we delve into the structure of the matrix A by decomposing it into its polar form. This entails a fundamental use of of the factorization $A = PU$ where P is a positive semidefinite matrix and U is an orthogonal matrix. This novel application of the polar decomposition is combined with the analysis of the evolution of the quantity $\int_{\mathbb{R}^2} u(x,t)(x^T Bx)dx$ for a well-chosen matrix B. This approach presents a novel modification of the method known in the literature as the "second moment technique for proving blow-up," where the key quantity is $\int_{\mathbb{R}^2} u(x,t)|x|^2 dx$. Lastly, we emphasize that our blow-up result encompasses, as particular cases, sharp results known for $A = I$, or A being a rotational matrix. Additionally, our research provides novel blow-up results, including instances when A is a positive definite matrix and when matrices are neither positive definite nor rotational, such as shear matrices. The research discussed in this chapter has been submitted for publication and is currently under review at the time of this thesis submission.

As mentioned previously, Chemotaxis is an intriguing biological phenomenon that plays a crucial role in enabling the aggregation and distribution of various species. It is a process that involves the movement of cells or organisms towards a chemical gradient, which is a concentration of molecules that stimulates the cells or organisms to move in a particular direction. Chemotaxis is an essential mechanism in many biological processes, including the immune response, wound healing, and embryonic development. It is also a critical factor in the behavior of microorganisms, such as bacteria, which use chemotaxis to locate nutrients and avoid toxins. Thus, the study of chemotaxis is essential to understanding the behavior and interactions of living organisms at the molecular level. This process involves the movement of organisms in response to a concentration gradient of chemicals. The model developed by Keller and Segel is widely recognized as a seminal contribution to the field of chemotaxis. It provides a mathematical framework for understanding the mechanisms underlying this complex biological process(e.g [52]). This model can be simplified by

$$\begin{aligned} u_t &= \Delta u - \chi\nabla\cdot(u\nabla v), \\ \varepsilon v_t &= \Delta v - v + u, \end{aligned} \tag{5.1}$$

where $u(x,t)$ denotes the density and $v(x,t)$ the chemical concentration at a given point x and time t. It is known for the case $\varepsilon = 0$ that in a two-dimensional domain setting the condition on the initial data $\int_{\mathbb{R}^2} u_0 dx < 8\pi/\chi$ implies the existence of global solutions meanwhile when $\int_{\mathbb{R}^2} u_0 > 8\pi/\chi$ the blow-up of solutions in finite time is possible (e.g. [15]).

An interesting variation of model (5.1) arise when taking into account that chemotactic migration in certain situations, are not necessarily parallel to the gradient of the signal. A key example is given by the dynamics of a type of bacteria known as peritrichously flagellated when swimming close to surfaces (e.g. [31, 92, 93]). In this case the evolution of the density of bacteria is describe by

$$u_t = \Delta u - \nabla\cdot(uA(x,u,v)\nabla v),$$

where the symbol $A(x,u,v)$ represents a 2×2 matrix. Several results of global existence and asymptotic behavior have been proved for this kind of models with tensorial chemotaxis during the last decade, see for instance [38, 87, 94] and the references therein. However, the possibility of having solutions blowing-up in finite time remains unclear when the chemoattractant is being produced by the cells itself. A first achievement in this direction, was reported for the parabolic-elliptic model

$$u_t = \Delta u - \chi\nabla\cdot(uA_\alpha\nabla v),\ x\in\mathbb{R}^2,\ t>0 \tag{5.2}$$

$$-\Delta v = u \tag{5.3}$$

where

$$A_a := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

denotes a rotation matrix with constant components. It was shown that blow-up of the solution in finite time is possible if and only if $\alpha\in(-\pi/2,\pi/2)$ and the initial data satisfies $\int u_0 dx > 8\pi/(\chi\cos\alpha)$. We refer the interested reader to [38] for details when working on the whole space and [94] for the corresponding analysis over on a bounded domain.

In this chapter, we aim to prove the possibility of having solutions blowing-up in a finite time for system

$$\begin{aligned} \partial_t u &= \Delta u - \chi\nabla\cdot(uA\nabla v), && x\in\mathbb{R}^2, t>0, \\ -\Delta v = u,\ v(x,t) &= -\tfrac{1}{2\pi}\int\log|x-y|\,u(y,t)dy && x\in\mathbb{R}^2, t>0, \\ u(x,0) &= u_0(x)\geq 0, && x\in\mathbb{R}^2, \end{aligned} \tag{5.4}$$

where $A := (a_{ij})_{i,j=1,2} \in M_2(\mathbb{R})$ represents a 2×2 matrix with constant components satisfying $Tr(A), \det(A) > 0$. Two interesting examples of matrices satisfying these hypothesis are the set of positive-definite matrices and the set of rotation matrices with angle $\alpha \in (-\pi/2, \pi/2)$. We also provide in the next section some examples of the possibility of having blow-up of solutions when the matrix A is neither orthogonal nor positive-definite. Our technique to prove blow-up is based on the polar decomposition of the matrix A and in a generalization of the well-known second moments technique, where instead of considering the evolution in time of the integral $\int u \, |x|^2 \, dx$, we analyze the evolution of the quantity $\int_{\mathbb{R}^2} u(x^T Bx)dx$ for some well chosen matrix B with constant components. Unlike the classical system with $A = I$, our criterion for deciding if a solution is blowing-up, will depend not only on the L^1-norm of the initial datum and the integrability of the function $x \to (x^T Bx)u(x,t)$ but also on

1. the determinant of A, denoted by $\det A$,
2. the trace of A, denoted by $Tr(A)$, and
3. the trace $Tr(\sqrt{AA^T})$, where A^T represents the transpose of the matrix A.

Here the symbol $\sqrt{AA^T}$ stands for the positive-definite square root of the matrix AA^T, whose existence and uniqueness is well-known in mathematics (cf. [74, Corollary 7.3.3]).

We summarize the main results of this chapter as follows.

Theorem 45 *Assume that the initial data u_0 satisfies $0 \le u_0 \in BUC(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, $\chi > 0$ is a constant and $A := (a_{ij})_{i,j=1,2}$ is an arbitrary matrix 2×2 with constant components. Then, there exists a maximal time $T_{\max} > 0$ of existence of a unique non-negative classical solution*

$$u \in C^0\left([0, T_{\max})\,; BUC(\mathbb{R}^2)\right) \cap C^0\left([0, T_{\max})\,; L^1(\mathbb{R}^2)\right) \cap C^\infty\left(\mathbb{R}^2 \times (0, T_{\max})\right),$$

to the system (5.4). Moreover, the quantity $\|u(\cdot,t)\|_{L^1(\mathbb{R}^2)}$ remains constant in time.
Let us define $\theta := \int_{\mathbb{R}^2} u_0 dx$. Then, we have

1. (Global existence) *for small enough mass θ, the system (5.4) has global-in-time solution, i.e., $T_{\max} = +\infty$,*
2. (Blow-up) if $Tr(A), \det(A) > 0$, $u_0 \, |x|^2 \in L^1(\mathbb{R}^2)$, and

$$\theta > \frac{4\pi}{\chi} \frac{\left(Tr\left((AA^T)^{1/2}\right)\right)^2}{Tr(A)\det(A)}$$

then $T_{\max} < +\infty$.

Corollary 46 ***(Positive-definite matrix case)*** *Consider a non-negative classical solution u of (5.4) with initial data $u_0 \geq 0$ satisfying $u_0 |x|^2 \in L^1(\mathbb{R}^2)$, and $A \in M_2(\mathbb{R})$ is a 2×2 positive-definite matrix (symmetric). Let $[0, T_{\max})$ be the maximal interval of existence and $\lambda_i > 0, i = 1, 2,$ the eigenvalues of A. If θ satisfies the condition*

$$\theta > \frac{4\pi}{\chi}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) = \frac{4\pi}{\chi}\frac{Tr(A)}{\det(A)}, \tag{5.5}$$

then, $T_{\max} < +\infty$.

5.1 Local existence, regularity, uniqueness, mass conservation and non-negativity for arbitrary matrices

Proposition 47 *Let $A \in M_2(\mathbb{R})$, and suppose that the initial data $u_0 \in BUC(\mathbb{R}^{\,2}) \cap L^1(\mathbb{R}^2)$ is non-negative. Then, there exist $T_{\max} \in (0, +\infty]$ and a non-negative*

$$u \in C^0\left([0, T_{\max}); BUC(\mathbb{R}^2)\right) \cap C^0\left([0, T_{\max}); L^1(\mathbb{R}^2)\right) \cap C^\infty\left(\mathbb{R}^2 \times (0, T_{\max})\right),$$

such that writing $v(\cdot, t) = \mathbf{K}(x) * u(\cdot, t), t \in (0, T_{\max})$, with $\mathbf{K}(x) := -\frac{1}{2\pi}\ln|x|, x \in \mathbb{R}^2 \backslash \{0\}$. we obtain $v \in C^\infty\left(\mathbb{R}^2 \times (0, T_{\max})\right)$,

$$\nabla v \in L^\infty_{loc}\left([0, T_{\max}); L^\infty(\mathbb{R}^2; \mathbb{R}^2)\right),$$

and that (u, v) forms a classical solution of (5.4) in $\mathbb{R}^2 \times (0, T_{\max})$. We also have the next extensibility criterion,

$$\text{if } T_{\max} < +\infty, \text{ then both } \limsup_{t \to T_{\max}} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} = +\infty,$$
$$\text{and } \limsup_{t \to T_{\max}} \||\nabla v(\cdot, t|)\|_{L^\infty(\mathbb{R}^2)} = +\infty.$$

This solution is uniquely determined in the sense that if $T \in (0, T_{\max})$, and if $(\widehat{u}, \widehat{v})$ is a classical solution of (5.4) in $\mathbb{R}^2 \times (0, T_{\max})$ fulfilling $\widehat{u} \in C^0([0, T]; BUC(\mathbb{R}^2)) \cap C^0([0, T]; L^1(\mathbb{R}^2)) \cap C^{2,1}(\mathbb{R}^2 \times (0, T))$ and $\widehat{v} \in C^{2,0}(\mathbb{R}^2 \times (0, T))$ as well as $\nabla \widehat{v} \in L^\infty(\mathbb{R}^2 \times (0, T); \mathbb{R}^2)$, then $\widehat{u} \equiv u$ in $\mathbb{R}^2 \times (0, T)$. Moreover,

$$\int_{\mathbb{R}^2} u(x, t)dx = \int_{\mathbb{R}^2} u_0 dx \text{ for all } t \in (0, T_{\max}). \tag{5.6}$$

Proof. See [87, Proposition 1.1.]. ■

5.2 Blow-up for the case $Tr(A)$, $det(A) > 0$

Theorem 48 (Blow-up) *Consider a non-negative classical solution u of system (5.4) with non-negative initial data $u_0 \in BUC(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and $u_0 |x|^2 \in L^1(\mathbb{R}^2)$. Suppose also that A is a 2×2 matrix with constant components satisfying $Tr(A), \det(A) > 0$. Let $[0, T_{\max})$ be the maximal interval of local existence of the solution guaranteed by Proposition 47. If θ satisfies the condition*

$$\theta > \tfrac{4\pi}{\chi}\frac{\left(Tr\left(\left(AA^T\right)^{1/2}\right)\right)^2}{Tr(A)\det(A)}, \tag{5.7}$$

then, $T_{\max} < +\infty$.

Proof. We start by decomposing the matrix A into the polar form

$$A = PU, \tag{5.8}$$

where $P = (p_{ij})_{i,j=1,2} := \left(AA^T\right)^{1/2}$ is positive-semidefinite and U is orthogonal (cf. [74, Corollary 7.3.3.]). The hypothesis $\det(A) > 0$ readily implies that P is positive-definite and $\det(U) = 1$. More precisely, we deduce that the orthogonal matrix U is a rotation matrix, thus it can be written in the form

$$U = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \text{ where } \alpha \in (-\pi, \pi].$$

On the other hand, the symmetry of the matrix P gives $p_{12} = p_{21}$ and henceforth the trace of the matrix A satisfies

$$\begin{aligned} Tr(A) &= Tr(PU) \\ &= Tr \begin{pmatrix} p_{11}\cos\alpha + p_{12}\sin\alpha & -p_{11}\sin\alpha + p_{12}\cos\alpha \\ p_{21}\cos\alpha + p_{22}\sin\alpha & -p_{21}\sin\alpha + p_{22}\cos\alpha \end{pmatrix} \\ &= \cos\alpha(p_{11} + p_{22}) \\ &= \cos\alpha Tr(P). \end{aligned}$$

It follows that

$$\cos\alpha = \frac{Tr(A)}{Tr(P)} > 0,$$

and therefore, we can assume without loss of generality that $\alpha \in (-\pi/2, \pi/2)$.

We now proceed to generalize the second moment blow-up technique (cf. [15]). With this end in mind, we multiply first the equation for the cell density u by the quadratic form $(x \cdot Bx)\,\varphi_R(x) \in C_0^\infty(\mathbb{R}^2)$, where B is a positive-definite matrix to be chosen later and $\varphi_R(x)$ is defined as in (3.72), which grows to $x \cdot Bx$ as $R \to \infty$. Next, we integrate the product to obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^2} u(x,t)\,(x\cdot Bx)\,\varphi_R(x)dx \\ &= \int_{\mathbb{R}^2}(x\cdot Bx)\,\varphi_R(x)\Delta u dx - \chi\int_{\mathbb{R}^2}(x\cdot Bx)\,\varphi_R(x)\nabla\cdot(uPU\nabla v)\,dx. \end{aligned}$$

Integration by parts gives then

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^2} u(x,t)\,(x\cdot Bx)\,\varphi_R(x)dx \\ &= \int_{\mathbb{R}^2}\Delta\left((x\cdot Bx)\,\varphi_R(x)\right)u dx + \chi\int_{\mathbb{R}^2}\nabla\left((x\cdot Bx)\,\varphi_R(x)\right)(uPU\nabla v)\,dx. \end{aligned}$$

We write explicitly the convolution $\nabla(\mathbf{K} * u)$ to get

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} u(x,t)\,(x\cdot Bx)\,\varphi_R(x)dx \\
&= \int_{\mathbb{R}^2} \Delta\left((x\cdot Bx)\,\varphi_R(x)\right)udx + \chi\int_{\mathbb{R}^2}\nabla\left((x\cdot Bx)\,\varphi_R(x)\right)\cdot\left(PU\nabla(\mathbf{K}*u)\right)udx \\
&= \int_{\mathbb{R}^2} \Delta\left((x\cdot Bx)\,\varphi_R(x)\right)udx \\
&+\chi\int_{\mathbb{R}^2}\nabla\left((x\cdot Bx)\,\varphi_R(x)\right)\cdot PU\left(\frac{-1}{2\pi}\int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}u(y,t)dy\right)u(x,t)dx \\
&= \int_{\mathbb{R}^2} \Delta\left((x\cdot Bx)\,\varphi_R(x)\right)udx \\
&-\frac{\chi}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\nabla\left((x\cdot Bx)\,\varphi_R(x)\right)\cdot PU\frac{x-y}{|x-y|^2}u(x,t)u(y,t)dy\right)dxdy. \qquad (5.9)
\end{aligned}
$$

We interchange x and y in the last integral to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\nabla\left((x\cdot Bx)\,\varphi_R(x)\right)\cdot PU\frac{x-y}{|x-y|^2}u(x,t)u(y,t)dy\right)dxdy \\
&= -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\nabla\left((y\cdot By)\,\varphi_R(y)\right)\cdot PU\frac{x-y}{|x-y|^2}u(x,t)u(y,t)dy\right)dx,
\end{aligned}
$$

which in turn implies

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\nabla\left((x\cdot Bx)\,\varphi_R(x)\right)\cdot PU\frac{x-y}{|x-y|^2}u(x,t)u(y,t)dy\right)dxdy \\
&= \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2}\left(\frac{[\nabla((x\cdot Bx)\varphi_R(x))-\nabla((y\cdot By)\varphi_R(y))]\cdot PU(x-y)}{|x-y|^2}u(x,t)u(y,t)dy\right)dx
\end{aligned}
$$

Thus, the identity (5.9) reduces to

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} u(x,t)\,(x\cdot Bx)\,\varphi_R(x)dx \\
&= \int_{\mathbb{R}^2} \Delta\left((x\cdot Bx)\,\varphi_R(x)\right)u(x,t)dx \\
&-\frac{\chi}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\left(\frac{[\nabla((x\cdot Bx)\varphi_R(x))-\nabla((y\cdot By)\varphi_R(y))]\cdot PU(x-y)}{|x-y|^2}u(x,t)u(y,t)dy\right)dx. \qquad (5.10)
\end{aligned}
$$

Because $\Delta\left((x\cdot Bx)\,\varphi_R(x)\right)$ remains bounded and $\nabla\left((x\cdot Bx)\,\varphi_R(x)\right)$ is Lipschitz continuous, we have that the two terms in the right-side of are bounded. Then, as $R\to\infty$, we can pass to the limit using the Lebesgue monotone convergence theorem with $u\in L^1(\mathbb{R}^2)$ in the integral version of (5.10) and thus

$$
\begin{aligned}
&\int_{\mathbb{R}^2} u\,(x\cdot Bx)\,dx - \int_{\mathbb{R}^2} u_0\,(x\cdot Bx)\,dx \\
&= \int_0^t\int_{\mathbb{R}^2}\Delta\,(x\cdot Bx)\,udxd\tau \\
&-\frac{\chi}{4\pi}\int_0^t\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{[\nabla((x\cdot Bx))-\nabla((y\cdot By))]\cdot PU(x-y)}{|x-y|^2}u(x,t)u(y,t)dydxd\tau.
\end{aligned}
$$

We notice that the symmetry of the matrix B gives the formula $\nabla (x \cdot Bx) = 2Bx$, and therefore

$$\begin{aligned}
&\int_{\mathbb{R}^2} u\,(x \cdot Bx)\,dx - \int_{\mathbb{R}^2} u_0\,(x \cdot Bx)\,dx \\
&= \int_0^t \int_{\mathbb{R}^2} \Delta\,(x \cdot Bx)\,udxd\tau \\
&- \frac{\chi}{2\pi} \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{B\,(x-y) \cdot PU\,(x-y)}{|x-y|^2} u(x,t)u(y,t)dydxd\tau.
\end{aligned}$$

Using the symmetry of the matrix B, we notice that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{B\,(x-y) \cdot PU\,(x-y)}{|x-y|^2} u(x,t)u(y,t)dydx \\
&= \int\limits_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{(x-y) \cdot BPU\,(x-y)}{|x-y|^2} u(x,t)u(y,t)dydx.
\end{aligned}$$

In consequence, we choose $B = P^{-1}$ to simplify the subsequent calculations. Then we get

$$\begin{aligned}
&\int_{\mathbb{R}^2} u\left(x \cdot P^{-1}x\right) dx - \int_{\mathbb{R}^2} u_0\left(x \cdot P^{-1}x\right) dx \\
&= \int_0^t \int_{\mathbb{R}^2} \Delta\left(x \cdot P^{-1}x\right) udxd\tau \\
&- \frac{\chi}{2\pi} \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{(x-y) \cdot U\,(x-y)}{|x-y|^2} u(x,t)u(y,t)dydxd\tau.
\end{aligned}$$

A simple computations also provide $\Delta\,(x \cdot P^{-1}x) = 2Tr(P^{-1}) = \frac{2Tr(P)}{\det(P)}$, thus we get

$$\begin{aligned}
&\int_{\mathbb{R}^2} u\left(x \cdot P^{-1}x\right) dx - \int_{\mathbb{R}^2} u_0\left(x \cdot P^{-1}x\right) dx \\
&= \frac{2Tr(P)}{\det(P)} \int_0^t \int_{\mathbb{R}^2} udxd\tau \\
&- \frac{\chi}{2\pi} \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{(x-y) \cdot U\,(x-y)}{|x-y|^2} u(x,t)u(y,t)dydxd\tau,
\end{aligned}$$

which can further be simplified using the mass conservation property (5.6) to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} u\left(x \cdot P^{-1}x\right) dx - \int_{\mathbb{R}^2} u_0\left(x \cdot P^{-1}x\right) dx \\
&= \frac{2Tr(P)}{\det(P)} \int_0^t \int_{\mathbb{R}^2} u_0 dxd\tau \\
&- \frac{\chi}{2\pi} \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{(x-y) \cdot U\,(x-y)}{|x-y|^2} u(x,t)u(y,t)dydxd\tau \\
&= \frac{2Tr(P)}{\det(P)} \theta t - \frac{\chi}{2\pi} \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{(x-y) \cdot U\,(x-y)}{|x-y|^2} u(x,t)u(y,t)dydxd\tau.
\end{aligned}$$

We now proceed to show that the orthogonality of matrix U allows for a significant reduction of the integral

$$\int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{(x-y)\cdot U\,(x-y)}{|x-y|^2} u(x,t)u(y,t)dydxd\tau.$$

To proceed with, we notice that for any $x \in \mathbb{R}^2$,

$$x^T \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} x = |x|^2 \cos\alpha,$$

and subsequently

$$\begin{aligned} &\int_{\mathbb{R}^2} u\left(x\cdot P^{-1}x\right)dx - \int_{\mathbb{R}^2} u_0\left(x\cdot P^{-1}x\right)dx \\ &= \frac{2Tr(P)}{\det(P)}\theta t - \frac{\chi\cos\alpha}{2\pi}\int_0^t\int_{\mathbb{R}^2\times\mathbb{R}^2} u(x,t)u(y,t)dydxd\tau. \end{aligned}$$

Using once again the mass conservation property, we arrive at the identity

$$\begin{aligned} &\int_{\mathbb{R}^2} u\left(x\cdot P^{-1}x\right)dx - \int_{\mathbb{R}^2} u_0\left(x\cdot P^{-1}x\right)dx \\ &= \frac{2Tr(P)}{\det(P)}\theta t - \frac{\chi\cos\alpha}{2\pi}\theta^2 t = \theta\left(\frac{2Tr(P)}{\det(P)} - \frac{\chi\cos\alpha}{2\pi}\theta\right)t. \end{aligned}$$

Thus

$$\begin{aligned} &\int_{\mathbb{R}^2} u(x,t)\left(x\cdot P^{-1}x\right)dx \\ &= \int_{\mathbb{R}^2} u(x,0)\left(x\cdot P^{-1}x\right)dx + \theta\left(\frac{2Tr(P)}{\det(P)} - \frac{\chi\cos\alpha}{2\pi}\theta\right)t. \end{aligned} \tag{5.11}$$

It follows from the hypothesis on the initial mass (5.7) that the right hand side of (5.11) will become negative a finite amount of time. On the other hand, the integral on the left hand side of (5.11) remains always positive due the nonnegativity of the variable u and the positive definiteness of the matrix P^{-1}. This contradiction implies $T_{\max} < \infty$. ■

Example 49 (Rotation matrix) *Let us consider the case*

$$A := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

with $\alpha \in (-\pi/2, \pi/2)$. Then Theorem 48 guarantees that the solutions of system (5.4) with initial data u_0 satisfying

$$\int_{\mathbb{R}^2} u_0(x)dx > \frac{4\pi}{\chi}\frac{\left(Tr\left(\left(AA^T\right)^{1/2}\right)\right)^2}{Tr(A)\det(A)} = \frac{8\pi}{\chi\cos\alpha},$$

will blow-up in a finite time. Notice that result coincides with the criterion of blow-up given in reference [38].

Example 50 (Horizontal shear matrix) *Let us consider the matrix*

$$A = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix},$$

which is known in fluid dynamics as the horizontal shear matrix with shear factor m. We notice that for $m \neq 0$, this matrix is neither positive definite nor orthogonal. However, in this case $Tr(A) = 2, \det(A) = 1 > 0$, which implies that Theorem 48 applies. In this case

$$AA^T = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} = \begin{pmatrix} m^2+1 & m \\ m & 1 \end{pmatrix},$$

which has as eigenvalues $\lambda_1 := \frac{1}{2}m\sqrt{m^2+4} + \frac{1}{2}m^2 + 1$ and $\lambda_2 := \frac{1}{2}m^2 - \frac{1}{2}m\sqrt{m^2+4} + 1$. It readily follows that

$$\begin{aligned} & Tr(\sqrt{AA^T}) \\ & = \sqrt{\lambda_1} + \sqrt{\lambda_2} \\ & = \frac{\sqrt{2m^2+4+2\sqrt{m^2(m^2+4)}} + \sqrt{2m^2+4-2\sqrt{m^2(m^2+4)}}}{2} \end{aligned}$$

Theorem 48 guarantees that the condition

$$\theta > \frac{\pi}{2\chi}\left(\sqrt{2m^2+4+2\sqrt{m^2(m^2+4)}} + \sqrt{2m^2+4-2\sqrt{m^2(m^2+4)}}\right)^2, \tag{5.12}$$

implies the corresponding solution u of system (5.4) blow-up in finite time. We notice in particular that for the case $m = 0$, we recover the well-known result of blow-up of solutions for the parabolic-elliptic Keller-Segel model (cf. [15]) getting that blow-up is feasible when $\theta > 8\pi/\chi$.

5.3 Global existence for small initial mass

Lemma 51 *Let $A := (a_{ij})_{i,j=1,2} \in M_2(\mathbb{R})$ and $p \in (1,\infty)$, one can find $\delta(p) > 0$ with the property that if $\theta \le \delta(p)$, and if $u_0 \in BUC(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ is non-negative, then there exists $C = C(p, u_0) > 0$ such that for the solution (u, v) of (5.4), as obtained in Proposition 47, we have*

$$\int_{\mathbb{R}^2} |u(x,t)|^p \, dx \le C \text{ for all } t \in (0, T_{\max}).$$

Proof. In a manner analogous to the approach in [87, p. 6], we select a smooth function $\zeta^{(0)} \in C^\infty(\mathbb{R})$ with the following properties: $0 \le \zeta^{(0)} \le 1$ on $\mathbb{R}$, $\zeta^{(0)} \equiv 1$ on $(-\infty, 0)$, and $\operatorname{supp}\zeta^{(0)} \subset (-\infty, 1)$. For $R > 1$, we define

$$\zeta_R(x) := \zeta^{(0)}(|x| - R), \quad x \in \mathbb{R}^n,$$

ensuring that $\zeta_R \in C_0^\infty(\mathbb{R}^n)$ with the conditions

$$0 \le \zeta_R \le 1 \text{ on } \mathbb{R}^n, \quad \zeta_R \equiv 1 \text{ in } B_R, \quad \text{and } \operatorname{supp}\zeta_R \subset B_{R+1},$$

where $B_R = B_R(0) \subset \mathbb{R}^n$ denotes the ball of radius $R > 1$ centered at the origin. Additionally, defining $K_\zeta := \|\zeta^{(0)}\|_{L^\infty(\mathbb{R})} + n\|\zeta^{(0)}\|_{L^\infty(\mathbb{R})}$, we observe that

$$|\nabla\zeta_R| + |\Delta\zeta_R| \le K_\zeta \text{ on } \mathbb{R}^n \text{ for all } R > 1. \tag{5.13}$$

To facilitate forthcoming estimations, we initiate by computing:

$$\begin{aligned}
&-\chi\int_{\mathbb{R}^2}\zeta_R^2u^{p-1}\nabla\cdot(uA\nabla v)dx\\
&=\chi\int_{\mathbb{R}^2}\nabla(\zeta_R^2u^{p-1})\cdot(uA\nabla v)dx\\
&=\chi\int_{\mathbb{R}^2}\big(2\zeta_R\nabla\zeta_Ru^{p-1}+(p-1)\zeta_R^2u^{p-2}\nabla u\big)\cdot(uA\nabla v)dx\\
&=\chi\int_{\mathbb{R}^2}\big(2\zeta_R\nabla\zeta_Ru^{p}+(p-1)\zeta_R^2u^{p-1}\nabla u\big)\cdot(A\nabla v)dx\\
&=\chi\int_{\mathbb{R}^2}\left(2\zeta_R\nabla\zeta_Ru^{p}+\frac{p-1}{p}\zeta_R^2\nabla u^p\right)\cdot(A\nabla v)dx\\
&=\chi\int_{\mathbb{R}^2}2\zeta_R\nabla\zeta_Ru^{p}\cdot(A\nabla v)dx+\frac{p-1}{p}\chi\int_{\mathbb{R}^2}\zeta_R^2\nabla u^p\cdot(A\nabla v)dx\\
&=\chi\int_{\mathbb{R}^2}2\zeta_R\nabla\zeta_Ru^{p}\cdot(A\nabla v)dx-\frac{p-1}{p}\chi\int_{\mathbb{R}^2}\zeta_R^2u^p\nabla\cdot(A\nabla v)dx\\
&-2\frac{p-1}{p}\chi\int_{\mathbb{R}^2}\zeta_R\nabla\zeta_Ru^{p}\cdot(A\nabla v)dx\\
&=2\chi(1-\frac{p-1}{p})\int_{\mathbb{R}^2}2\zeta_R\nabla\zeta_Ru^{p}\cdot(A\nabla v)dx-\frac{p-1}{p}\chi\int_{\mathbb{R}^2}\zeta_R^2u^p\nabla\cdot(A\nabla v)dx\\
&=\frac{2\chi}{p}\int_{\mathbb{R}^2}\zeta_R\nabla\zeta_Ru^{p}\cdot(A\nabla v)dx-\frac{p-1}{p}\chi\int_{\mathbb{R}^2}\zeta_R^2u^p\nabla\cdot(A\nabla v)dx.
\end{aligned}\tag{5.14}$$

Next, we multiply the equation for the cell density u in system (5.4) by $p\zeta_R^2u^{p-1}$, integrate by parts, and apply identity (5.14) to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}\zeta_R^2u^pdx+p(p-1)\int_{\mathbb{R}^2}\zeta_R^2u^{p-2}|\nabla u|^2dx\\
&=-2p\int_{\mathbb{R}^2}\zeta_Ru^{p-1}\nabla u\cdot\nabla\zeta_Rdx\\
&+\frac{2\chi}{p}\int_{\mathbb{R}^2}\zeta_Ru^p\nabla\zeta_R\cdot(A\nabla v)dx-\frac{p-1}{p}\chi\int_{\mathbb{R}^2}\zeta_R^2u^p\nabla\cdot(A\nabla v)dx.
\end{aligned}\tag{5.15}$$

for all $t \in (0, T_{\max})$. To estimate the first term on the right-hand side (5.15), we utilize Young´s inequality and (5.13) in the form

$$\begin{aligned}
&-2p\int_{\mathbb{R}^2}\zeta_Ru^{p-1}\nabla u\cdot\nabla\zeta_Rdx \qquad (5.16)\\
&\le\frac{p(p-1)}{2}\int_{\mathbb{R}^2}\zeta_R^2u^{p-2}|\nabla u|^2dx+\frac{2pK_\zeta^2}{p-1}\int_{B_{R+1}\setminus B_R}u^pdx \quad \text{for all } t\in(0,T_{\max}).
\end{aligned}$$

To estimate the second term on the right-hand side in (5.15), we first notice that

$$c(T) := \sup_{t\in(0,T)}\|\nabla v(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}$$

is finite for each $T \in (0, T_{\max})$ due to Proposition 47. Hence

$$\frac{2\chi}{p}\int_{\mathbb{R}^2} \zeta_R u^p \nabla\zeta_R \cdot (A\nabla v)dx \leq \frac{2\chi K_\zeta \left|A\right| c(T)}{p}\int_{B_{R+1}\setminus B_R} u^p dx \quad \text{for all } t \in (0,T). \tag{5.17}$$

The last term in (5.15) is estimate as follows:

$$\begin{aligned}
&-\frac{p-1}{p}\chi\int_{\mathbb{R}^2} \zeta_R^2 u^p \nabla\cdot(A\nabla v)dx \\
&\leq \frac{(p-1)\chi}{p+1}\int_{\mathbb{R}^2} \zeta_R^2 u^{p+1}dx + \frac{(p-1)\chi}{p(p+1)}\int_{\mathbb{R}^2} \zeta_R^2 \left|\nabla\cdot(A\nabla v)\right|^{p+1} dx \\
&\leq \frac{(p-1)\chi}{p+1}\int_{\mathbb{R}^2} u^{p+1}dx + \frac{(p-1)\left|A\right|^{p+1}\chi}{p(p+1)}\int_{\mathbb{R}^2} \left|D^2 v\right|^{p+1} dx.
\end{aligned}$$

Now, we apply the Calderón–Zygmund inequality (See for instance [45, Section 6.4.2.]) to obtain a constant c_1 such that

$$\int_{\mathbb{R}^2} \left|D^2 v\right|^{p+1} dx \leq c_1 \int_{\mathbb{R}^2} u^{p+1}dx,$$

thus

$$\begin{aligned}
&\frac{p-1}{p}\chi\int_{\mathbb{R}^2} \zeta_R^2 u^p \nabla\cdot(A\nabla v)dx \\
&\leq \frac{(p-1)\chi}{p+1}\int_{\mathbb{R}^2} u^{p+1}dx + \frac{(p-1)\left|A\right|^{p+1}\chi c_1}{p(p+1)}\int_{\mathbb{R}^2} u^{p+1}dx \\
&= \frac{(p-1)\chi\left(p+\left|A\right|^{p+1}c_1\right)}{p(p+1)}\int_{\mathbb{R}^2} u^{p+1}dx.
\end{aligned} \tag{5.18}$$

To estimate $\int_{\mathbb{R}^2} u^{p+1}$, we utilize the classical Gagliardo-Nirenberg-Sobolev inequality: For $1 \leq q < n$, it holds

$$\left\|w\right\|_{L^{q^*}} \leq c_2 \left\|\nabla w\right\|_{L^q}, \text{ for all } w \in W^{1,q}(R^n).$$

where c_2 is a constant and q^* is given by $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n}$. In particular for $q = 1$, $n = 2$, we obtain

$$\int_{\mathbb{R}^2} w^2 dx \leq c_2^2 \left(\int_{\mathbb{R}^2} \left|\nabla w\right| dx\right)^2.$$

Taking $u^{p+1} = w^2$ yields

$$\begin{aligned}
\int_{\mathbb{R}^2} u^{p+1}dx &\leq c_2^2\left(\int_{\mathbb{R}^2}\left|\nabla u^{\frac{p+1}{2}}\right|dx\right)^2 \\
&= c_2^2\left(\frac{p+1}{p}\right)^2\left(\int_{\mathbb{R}^2}\left|u^{1/2}\nabla\left(u^{p/2}\right)\right|dx\right)^2 \\
&\leq c_2^2\left(1+\frac{1}{p}\right)^2\theta\int_{\mathbb{R}^2}\left|\nabla\left(u^{p/2}\right)\right|^2 dx.
\end{aligned} \tag{5.19}$$

In conclude from (5.18) and (5.19)

$$\begin{aligned}
&\frac{p-1}{p}\chi\int_{\mathbb{R}^2}\zeta_R^2 u^p\nabla\cdot(A\nabla v)dx \\
&\le \frac{(p-1)\chi\left(p+|A|^{p+1}c_1\right)}{p(p+1)}c_2^2\left(1+\frac{1}{p}\right)^2\theta\int_{\mathbb{R}^2}\left|\nabla\left(u^{p/2}\right)\right|^2dx.
\end{aligned} \tag{5.20}$$

In summary, the differential inequality(5.15),(5.16) together with the estimates (5.17) and (5.20), imply that for $T\in(0,T_{\max})$

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}\zeta_R^2u^pdx+p(p-1)\int_{\mathbb{R}^2}\zeta_R^2u^{p-2}|\nabla u|^2dx \\
&\le \frac{p(p-1)}{2}\int_{\mathbb{R}^2}\zeta_R^2u^{p-2}|\nabla u|^2dx+\frac{2pK_\zeta^2}{p-1}\int_{B_{R+1}\setminus B_R}u^pdx \\
&+\frac{2\chi K_\zeta\,|A|\,c(T)}{p}\int_{B_{R+1}\setminus B_R}u^pdx \\
&+\frac{(p-1)\chi\left(p+|A|^{p+1}c_1\right)}{p(p+1)}c_2^2\left(1+\frac{1}{p}\right)^2\theta\int_{\mathbb{R}^2}\left|\nabla\left(u^{p/2}\right)\right|^2dx.
\end{aligned}$$

Equivalently, using the identity $\int_{\mathbb{R}^2}\zeta_R^2u^{p-2}|\nabla u|^2dx=\frac{2}{p}\int_{\mathbb{R}^2}\zeta_R^2\left|\nabla u^{p/2}\right|^2dx$,

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}\zeta_R^2u^pdx \\
&+\left(p-1-\tfrac{(p-1)\chi\left(p+|A|^{p+1}c_1\right)}{p(p+1)}C_{GNS}^2\left(1+\frac{1}{p}\right)^2\theta\right)\int_{\mathbb{R}^2}\zeta_R^2\left|\nabla u^{p/2}\right|^2dx \\
&\le\left(\frac{pK_\zeta^2}{p-1}+\frac{2\chi K_\zeta\,|A|\,c(T)}{p}\right)\int_{B_{R+1}\setminus B_R}u^pdx \\
&+\frac{(p-1)\chi\left(p+|A|^{p+1}c_1\right)}{p(p+1)}c_2^2\left(1+\frac{1}{p}\right)^2\theta\int_{\mathbb{R}^2}(1-\zeta_R^2)\left|\nabla\left(u^{p/2}\right)\right|^2dx.
\end{aligned}$$

Taking θ small enough to satisfy

$$p-1-\frac{(p-1)\chi\left(p+|A|^{p+1}c_1\right)}{p(p+1)}c_2^2\left(1+\frac{1}{p}\right)^2\theta>0,$$

we derive

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}\zeta_R^2u^pdx\le\left(\frac{pK_\zeta^2}{p-1}+\frac{2\chi K_\zeta\,|A|\,c(T)}{p}\right)\int_{B_{R+1}\setminus B_R}u^pdx \\
&+\frac{(p-1)\chi\left(p+|A|^{p+1}c_1\right)}{p(p+1)}c_2^2\left(1+\frac{1}{p}\right)^2\theta\int_{\mathbb{R}^2}(1-\zeta_R^2)\left|\nabla\left(u^{p/2}\right)\right|^2dx,
\end{aligned}$$

so that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\zeta_R^2u^pdx \\
&\le\int_{\mathbb{R}^2}\zeta_R^2u^p(x,0)+\left(\frac{pK_\zeta^2}{p-1}+\frac{2\chi K_\zeta\,|A|\,c(T)}{p}\right)\int_0^t\int_{B_{R+1}\setminus B_R}u^pdxd\tau \\
&+\frac{(p-1)\chi\left(p+|A|^{p+1}c_1\right)}{p(p+1)}c_2^2\left(1+\frac{1}{p}\right)^2\theta\int_0^t\int_{\mathbb{R}^2}(1-\zeta_R^2)\left|\nabla\left(u^{p/2}\right)\right|^2dxd\tau.
\end{aligned} \tag{5.21}$$

since $u^p(x.t) \leq |u|^{p-1}_{L^\infty(\mathbb{R}^2)} u(x,t)$ for all $(x,t) \in \mathbb{R}^2 \times (0,T)$, we conclude by dominated convergence

$$\int_0^t \int_{B_{R+1}\setminus B_R} u^p dx d\tau \to 0 \text{ as } R \to \infty \text{ for all } t \in (0,T). \tag{5.22}$$

On the other hand, Beppo Levi's theorem implies

$$\int_0^t \int_{\mathbb{R}^2} \zeta_R^2 \left|\nabla \left(u^{p/2}\right)\right|^2 dx d\tau \nearrow \int_0^t \int_{\mathbb{R}^2} \left|\nabla \left(u^{p/2}\right)\right|^2 dx d\tau, \tag{5.23}$$

and

$$\int_{\mathbb{R}^2} \zeta_R^2 u^p(x,t) dx \nearrow \int_{\mathbb{R}^2} u^p(x,t) dx, \tag{5.24}$$

for all $t \in (0,T)$.From the estimate (5.21), together with the limits (5.22)-(5.24), we conclude

$$\int_{\mathbb{R}^2} u^p(x,t) dx \leq \int_{\mathbb{R}^2} u^p(x,0) dx, \text{ for all } t \in (0,T).$$

Since $T \in (0, T_{\max})$ was arbitrary, this establishes the claim. ■

Theorem 52 (Global existence) *Let $A := (a_{ij})_{i,j=1,2} \in M_2(\mathbb{R})$. Then, there exists $\delta > 0$ with the property that if $\theta \leq \delta$, for any non-negative $u_0 \in BUC(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, the solution (u,v) of (5.4) is global and bounded; that is, in Proposition 47 we have $T_{\max} = +\infty$, and there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \leq C \text{ for all } t > 0.$$

Proof. The proof follows straightforward from Lemma 51 and [87, Lemma 4.2], starting at the inequality (4.10) of that paper. ■

Chapter 6

Blow-up of solutions to the Keller-Segel model with tensorial flux in high dimensions

Abstract

In recent years, there has been a notable upsurge in the examination of Keller-Segel models incorporating tensorial flux. Despite this interest, the question of whether finite-time blowup solutions exist remains a topic of ongoing research. In this chapter, we provide evidence that solutions of this nature are indeed possible in dimensions $n \geq 3$, when utilizing a tensorial flux expressed in the form of Av, where A denotes a matrix with constant components. The research discussed in this chapter has been accepted for publication in the journal *Applied Mathematics Letters* (Volume 154, August 2024, 109090) under the title: *Blow-up of solutions to the Keller–Segel model with tensorial flux in high dimensions.*

As introduced in the previous chapter, there has been a significant level of interest in the analysis of Keller-Segel models incorporating tensorial flux over the past decade. Despite this interest, the question of whether finite-time blowup solutions exist remains a topic of ongoing research. In chapter 5, we aimed to demonstrate the possibility of finite-time blowup solutions in the two-dimensional Keller-Segel model when having a tensorial flux of the form $A\nabla v$, where A represents an arbitrary 2×2 matrix with constant components satisfying $Tr(A)$ and $\det(A)$ both being greater than zero. Building on this, our objective in this chapter is to provide compelling evidence that such solutions are indeed possible in higher dimensions, when utilizing a tensorial flux expressed as $A\nabla v$, where A represents a matrix with constant components and satisfies quite general conditions.

Specifically, we aim to prove the possibility of having solutions blowing-up in a finite time for system

$$\begin{array}{rr} \partial_t u = \Delta u - \chi \nabla \cdot (uA\nabla v), & x \in \mathbb{R}^n, t > 0, \\ -\Delta v = u,\ v(x,t) = \frac{1}{n(n-2)|B_1(0)|} \int u(y,t)\, |x-y|^{2-n}\, dy & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^n, \end{array} \tag{6.1}$$

where $A := (a_{ij})_{i,j=1,\dots,n} \in M_n(\mathbb{R})$ represents a nonsingular $n \times n$ matrix with constant components satisfying $x^T \left(\left(AA^T \right)^{1/2} \right)^{-1} Ax > 0$ for all non-zero $x \in \mathbb{R}^n$. Here the symbol $\sqrt{AA^T}$ stands for the positive-definite square root of the matrix AA^T, whose existence and uniqueness is well-established in mathematics (c.f. [74, Corollary 7.3.3]). Examples of matrices satisfying this hypothesis include the set of positive-definite matrices and, in the three-dimensional case, orthogonal matrices of the form

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\alpha \in (-\pi/2, \pi/2)$. Our approach to proving blow-up involves decomposing matrix A into its polar components and employing a modified version of the second moments technique. In contrast to the nontensorial Keller-Segel model, where the evolution of $\int_{\mathbb{R}^n} u(x,t) \, |x|^2 \, dx$ is fundamental, we reveal that the tensorial attraction makes $\int_{\mathbb{R}^n} u(x,t)(x^T Bx)dx$ crucial, where the matrix B, with constant component, is meticulously chosen to yield the desired outcome of blow-up.

6.1 Local existence, regularity, uniqueness, mass conservation and non-negativity for arbitrary matrices

Proposition 53 *Let $n \geq 3$ and $A \in M_n(\mathbb{R})$, and suppose that the initial data $u_0 \in BUC(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ is non-negative. Then, there exist $T_{\max} \in (0, +\infty]$ and a non-negative*

$$u \in C^0 \left([0, T_{\max}) ; BUC(\mathbb{R}^n) \right) \cap C^0 \left([0, T_{\max}) ; L^1(\mathbb{R}^n) \right) \cap C^\infty \left(\mathbb{R}^n \times (0, T_{\max}) \right),$$

such that writing $v(\cdot,t) = \mathbf{K}(x) * u(\cdot,t), t \in (0, T_{\max})$, with $\mathbf{K}(x) := \frac{1}{n(n-2)|B_1(0)|} \, |x|^{2-n}, x \in \mathbb{R}^n \backslash \{0\}$. we obtain $v \in C^\infty \left(\mathbb{R}^n \times (0, T_{\max}) \right)$, $\nabla v \in L^\infty_{loc}([0, T_{\max}) ; L^\infty \left(\mathbb{R}^n ; \mathbb{R}^n \right))$, and that (u, v) forms a classical solution of (6.1) in $\mathbb{R}^n \times (0, T_{\max})$. We also have the next extensibility criterion,

$$\text{if } T_{\max} < +\infty, \text{ then both } \limsup_{t \to T_{\max}} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = +\infty$$
$$\text{and } \limsup_{t \to T_{\max}} \|\nabla v(\cdot,t))\|_{L^\infty(\mathbb{R}^n)} = +\infty.$$

This solution is uniquely determined in the sense that if $T \in (0, T_{\max})$, and if $(\widehat{u}, \widehat{v})$ is a classical solution of (6.1) in $\mathbb{R}^n \times (0, T_{\max})$ fulfilling $\widehat{u} \in C^0([0,T] ; BUC \ (\mathbb{R}^n)) \cap C^0 \left([0,T] ; L^1(\mathbb{R}^n) \right) \cap C^{2,1} \left(\mathbb{R}^n \times (0,T) \right)$ and $\widehat{v} \in C^{2,0} \left(\mathbb{R}^n \times (0,T) \right)$ as well as $\nabla \widehat{v} \in L^\infty \left(\mathbb{R}^n \times (0,T) ; \mathbb{R}^n \right)$, then $\widehat{u} \equiv u$ in $\mathbb{R}^n \times (0,T)$. Moreover,

$$\int_{\mathbb{R}^n} u(x,t)dx = \int_{\mathbb{R}^n} u_0 dx =: M \text{ for all } t \in (0, T_{\max}). \tag{6.2}$$

Proof. See [87, Proposition 1.1.]. ■

6.2 Blow-up

Our methodology to establish blow-up in high dimensions hinges upon the technique proposed in Chapter 5 for the analysis of blow-up for two-dimensional Keller-Segel type systems with tensorial flux. This methodology can be outlined in two key steps: firstly, leveraging the polar decomposition of the tensor A and secondly, examining the evolution of the quantity $\int u(x^T Bx)dx$ using a strategically chosen matrix B with constant components.

Theorem 54 (Blow-up) *Given $n \geq 3$, consider a non-negative classical solution u of system (6.1) with non-negative initial data $u_0 \in BUC(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $u_0 |x|^2 \in L^1(\mathbb{R}^n)$. Suppose also that $A \in M_n(\mathbb{R})$ is a nonsingular matrix with constant components satisfying*

$$x^T \left(\left(AA^T\right)^{1/2} \right)^{-1} Ax > 0 \text{ for all non-zero } x \in \mathbb{R}^n. \tag{6.3}$$

Let $[0, T_{\max})$ be the maximal interval of local existence of the solution guaranteed by Proposition 53. If the integral $m_0 := \int_{\mathbb{R}^n} u_0 |x|^2 dx$ is small enough compared to the mass M, more precisely, if for a constant $C_{Bl} := C(A, \chi, n) > 0$

$$\int_{\mathbb{R}^n} u_0 |x|^2 dx \leq C_{Bl} M^{\frac{n}{n-2}}, \tag{6.4}$$

then $T_{\max} < +\infty$.

Proof. To facilitate the presentation, we conduct a formal calculation of the evolution of moments, assuming the solution u is suitably regular and decay sufficiently fast at infinity. We start by decomposing the nonsingular matrix A into the polar form $A = PU$, where $P = (p_{ij})_{i,j=1,n} := \left(AA^T\right)^{1/2}$ is positive-definite and $U := P^{-1}A$ is orthogonal (cf. [74, Corollary 7.3.3.]). Next, we proceed to modify the second-moment blow-up technique by multiplying the equation for the cell density u by the quadratic form $x \cdot Bx$, where B is a positive definite matrix to be determined. Integrating the product, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^n} u\,(x \cdot Bx)\, dx = \int_{\mathbb{R}^n} (x \cdot Bx)\, \Delta u dx - \chi \int_{\mathbb{R}^n} (x \cdot Bx)\, \nabla \cdot (uPU\nabla v)\, dx.$$

Integration by parts leads to

$$\frac{d}{dt} \int_{\mathbb{R}^n} u\,(x \cdot Bx)\, dx = \int_{\mathbb{R}^n} \Delta\,(x \cdot Bx)\, u dx + \chi \int_{\mathbb{R}^n} \nabla\,(x \cdot Bx)\,(uPU\nabla v)\, dx.$$

Considering the symmetry of the matrix B, the formula $\nabla\,(x \cdot Bx) = 2Bx$ holds, and therefore

$$\frac{d}{dt} \int_{\mathbb{R}^n} u\,(x \cdot Bx)\, dx = \int_{\mathbb{R}^n} \Delta\,(x \cdot Bx)\, u dx + \chi \int_{\mathbb{R}^n} 2Bx \cdot (uPU\nabla v)\, dx.$$

Utilizing again the symmetry of the matrix B, the last integral can be rewritten as

$$\int_{\mathbb{R}^n} 2Bx \cdot (uPU\nabla v)\, dx = 2 \int_{\mathbb{R}^n} x \cdot (BPU\nabla v)\, u dx.$$

Consequently, we choose $B = P^{-1}$ to simplify the subsequent calculations. This leads to

$$\frac{d}{dt}\int_{\mathbb{R}^n} u\left(x\cdot P^{-1}x\right)dx = \int_{\mathbb{R}^n}\Delta\left(x\cdot P^{-1}x\right)udx + 2\chi\int_{\mathbb{R}^n} x\cdot(U\nabla v)\,udx.$$

Direct computations yield $\Delta\left(x\cdot P^{-1}x\right) = 2Tr(P^{-1})$. Thus

$$\frac{d}{dt}\int_{\mathbb{R}^n} u\left(x\cdot P^{-1}x\right)dx = 2Tr(P^{-1})\int_{\mathbb{R}^n} udx + 2\chi\int_{\mathbb{R}^n} x\cdot(U\nabla(\mathbf{K}*u))\,udx.$$

This expression can be further simplified using the mass conservation property (6.2) to obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n} u\left(x\cdot P^{-1}x\right)dx &= 2Tr(P^{-1})\int_{\mathbb{R}^n} u_0dx + 2\chi\int_{\mathbb{R}^n} x\cdot(U\nabla(\mathbf{K}*u))\,udx\\ &= 2Tr(P^{-1})M + 2\chi\int_{\mathbb{R}^n} x\cdot(U\nabla(\mathbf{K}*u))\,udx.\end{aligned}$$

We now proceed to show that the orthogonality of matrix U allows for a significant reduction of the integral $\int_{\mathbb{R}^n} x\cdot(uU\nabla(\mathbf{K}*u))\,dx$. First, we explicitly write the convolution $\nabla(\mathbf{K}*u)$ to get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n} u\left(x\cdot P^{-1}x\right)dx\\ &= 2Tr(P^{-1})M + 2\chi\int_{\mathbb{R}^n} x\cdot(U\nabla(\mathbf{K}*u))\,udx\\ &= 2Tr(P^{-1})M + 2\chi\int_{\mathbb{R}^n} x\cdot U\left(\frac{-1}{n\,|B_1(0)|}\int_{\mathbb{R}^n}\frac{x-y}{|x-y|^n}u(y,t)dy\right)u(x,t)dx\\ &= 2Tr(P^{-1})M - \frac{2\chi}{n\,|B_1(0)|}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(x\cdot U\frac{x-y}{|x-y|^n}u(x,t)u(y,t)dy\right)dxdy.\end{aligned} \tag{6.5}$$

We interchange x and y in the last integral to obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(x\cdot U\frac{x-y}{|x-y|^n}u(x,t)u(y,t)dy\right)dxdy\\ &= -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(y\cdot U\frac{x-y}{|x-y|^n}u(x,t)u(y,t)dy\right)dx,\end{aligned}$$

which implies

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(x\cdot U\frac{x-y}{|x-y|^n}u(x,t)u(y,t)dy\right)dxdy\\ &= \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}\left((x-y)\cdot U\frac{x-y}{|x-y|^n}u(x,t)u(y,t)dy\right)dx.\end{aligned}$$

Thus, the identity (6.5) reduces to

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n} u\left(x\cdot P^{-1}x\right)dx\\ &= 2Tr(P^{-1})M - \frac{\chi}{n\,|B_1(0)|}\int_{\mathbb{R}^n\times\mathbb{R}^n}\left((x-y)\cdot U\frac{x-y}{|x-y|^n}u(x,t)u(y,t)dy\right)dx.\end{aligned}$$

Next, we observe that since U is an orthogonal matrix, there is an orthogonal matrix Q and a block diagonal matrix D such that

$$QUQ^T = D = \begin{pmatrix} R_1 & & & & & \\ & \ddots & & & \mathbf{0} & \\ & & R_k & & & \\ & & & \lambda_1 & & \\ & \mathbf{0} & & & \ddots & \\ & & & & & \lambda_p \end{pmatrix}, \tag{6.6}$$

where all the R_j represent a 2×2 rotation matrix (cf. [74, Corollary 2.5.14. (c)]), that is a matrix of the form

$$R_j = \begin{pmatrix} \cos\alpha_j & -\sin\alpha_j \\ \sin\alpha_j & \cos\alpha_j \end{pmatrix}, \text{ where } \alpha_j \in (-\pi, \pi],$$

and each λ_j can take solely the values 1 or -1. Moreover, the hypothesis that $0 < x^T \left(\left(AA^T\right)^{1/2} \right)^{-1} Ax = x^T P^{-1} Ax = x^T U x$ for all non-zero $x \in \mathbb{R}^n$, readily implies that $\lambda_i = 1, i = 1, \ldots, p$, and $\cos\alpha_j > 0, j = 1, \ldots, k$.Therefore, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} x^T U x = x^T Q^T D Q x &= (Qx)^T D (Qx) = (Qx)^T \left(\frac{1}{2}\left(D + D^T\right) \right)(Qx) \\ &= (Qx)^T \begin{pmatrix} \cos\alpha_1 & & & & & & & \\ & \cos\alpha_1 & & & & & & \mathbf{0} \\ & & \ddots & & & & & \\ & & & \cos\alpha_k & & & & \\ & & & & \cos\alpha_k & & & \\ & & & & & 1 & & \\ & \mathbf{0} & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} (Qx) \\ &\geq \min_{j=1,\ldots,k}\{\cos\alpha_j, 1\}(Qx)^T(Qx) = \min_{j=1,\ldots,k}\{\cos\alpha_j, 1\}\left|Ox\right|^2 \\ &= \min_{j=1,\ldots,k}\{\cos\alpha_j, 1\}\left|x\right|^2, \end{aligned}$$

and subsequently

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n} u\left(x \cdot P^{-1}x\right) dx \\ &\leq 2Tr(P^{-1})M - \frac{\chi \min_{j=1,\ldots,k}\{\cos\alpha_j, 1\}}{n\left|B_1(0)\right|} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{1}{|x-y|^{n-2}} u(x,t)u(y,t)dydx. \end{aligned}$$

To simplify the last inequality, we invoke a result from [12, Lemma 3.2.], which states that for any nonnegative function $f \in L^1(\mathbb{R}^n, (1 + |x|^2)dx)$, the moment $m = \int_{\mathbb{R}^n} f(x)\left|x\right|^2 dx$, the mass $M = \int_{\mathbb{R}^n} f(x)dx$ and the integral $J := \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x)f(y)\left|x-y\right|^{2-n} dydx$, satisfy the inequality $M^{\frac{n}{2}+1} \leq J(2m)^{\frac{n}{2}-1}$.

Therefore

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{1}{|x-y|^{n-2}} u(x,t)u(y,t)dydx \geq M^{\frac{n}{2}+1}\left(2\int_{\mathbb{R}^n} u\left|x\right|^2 dx\right)^{1-\frac{n}{2}}. \tag{6.7}$$

and the functions $w(t) := \int u(x,t)\,(x\cdot P^{-1}x)\,dx$ and $m(t) := \int u(x,t)\,|x|^2\,dx$ satisfy

$$\frac{d}{dt}w(t) \le 2Tr(P^{-1})M - \frac{2^{1-\frac{n}{2}}\chi \min_{j=1,\dots,k}\{\cos\alpha_j, 1\}M^{\frac{n}{2}+1}}{n\,|B_1(0)|}\,(m(t))^{1-\frac{n}{2}}. \quad (6.8)$$

Let us denote the minimum and maximum eigenvalues of P^{-1} by $\lambda_{\min}$ and $\lambda_{\max}$, respectively. A standard result (cf. [74, Theorem 4.2.2.]) asserts $\lambda_{\min}|x|^2 \le x^T P^{-1}x \le \lambda_{\max}|x|^2$ for all $x \in \mathbb{R}^n$, yielding $\lambda_{\min}m(t) \le w(t) \le \lambda_{\max}m(t)$ for all $t \ge 0$, and

$$\frac{d}{dt}w(t) \le 2Tr(P^{-1})M - \frac{2^{1-\frac{n}{2}}\chi \min_{j=1,\dots,k}\{\cos\alpha_j, 1\}M^{\frac{n}{2}+1}}{n\,|B_1(0)|}\,(\lambda_{\min})^{\frac{n}{2}-1}\,(w(t))^{1-\frac{n}{2}}. \quad (6.9)$$

This reads as the differential inequality

$$\frac{2}{n}\frac{d}{dt}w^{n/2} \le 2Tr(P^{-1})Mw^{\frac{n}{2}-1} - \frac{2^{1-\frac{n}{2}}\chi \min_{j=1,\dots,k}\{\cos\alpha_j, 1\}M^{\frac{n}{2}+1}}{n\,|B_1(0)|}\,(\lambda_{\min})^{\frac{n}{2}-1} =: f(w). \quad (6.10)$$

We now introduce the condition on the initial data $f(w(0)) < 0$. Since f is an increasing function of w, the condition $f(w(0)) < 0$ implies that the right-hand side of (6.10) is always negative and bounded away from zero. We conclude that the right hand side is always negative and bounded away from zero. This leads to the conclusion that the function w decreases and assumes negative values in a finite time, contradicting the existence of a global in time nonnegative solution. Finally, observing the inequality $w(t) \le \lambda_{\max}m(t)$, we obtain

$$f(w) \le 2Tr(P^{-1})M\lambda_{\max}^{\frac{n}{2}-1}m^{\frac{n}{2}-1} - \frac{2^{1-\frac{n}{2}}\chi \min_{j=1,\dots,k}\{\cos\alpha_j, 1\}M^{\frac{n}{2}+1}}{n\,|B_1(0)|}\,(\lambda_{\min})^{\frac{n}{2}-1} =: h(m).$$

Hence the condition on the initial moment $h(m(0))) < 0$ or equivalently

$$\int_{\mathbb{R}^n} u(x,0)\,|x|^2\,dx \le \left(\frac{2^{1-\frac{n}{2}}\chi \min_{j=1,\dots,k}\{\cos\alpha_j, 1\}}{2Tr(P^{-1})\lambda_{\max}^{\frac{n}{2}-1}n\,|B_1(0)|}\,(\lambda_{\min})^{\frac{n}{2}-1}\right)^{\frac{2}{n-2}} M^{\frac{n}{n-2}},$$

implies that $T_{\max} < \infty$. ■

Remark 55 *For all $M > 0$ (even arbitrarily small), there exists an initial data u_0 with mass M such that the condition (6.4) is satisfied. Indeed, it is sufficient to consider non-negative, smooth, compactly supported data u_0 with mass M and second moment m_0. By rescaling it with $\varepsilon^{-n}u_0\left(\frac{x}{\varepsilon}\right)$ for a sufficiently small $\varepsilon > 0$ (specifically, $\varepsilon^2 \le \frac{C_{Bl}M^{\frac{n}{n-2}}}{m_0}$), the desired condition is achieved. In other words, blow-up is still possible for arbitrarily small initial mass, which contrasts with the two-dimensional case (see Chapter 5).*

6.3 Global existence

Theorem 56 (Global existence) *Let $A := (a_{ij})_{i,j=1,\dots,n} \in M_n(\mathbb{R})$ be a matrix with constant components. Then, there exists $\delta > 0$ with the property that if $\|u_0\|_{L^{\frac{n}{2}}} \le \delta$, for any non-negative $u_0 \in BUC(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, the solution u of the system (6.1) is global and for some constant $C > 0$, we have that $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le C$ for all $t > 0$.*

Proof. By multiplying the equation for u by u^{p-1} and integrating over $\mathbb{R}^n$, we derive

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^n} |u(x,t)|^p\,dx\\ &= -\frac{4(p-1)}{p^2}\int_{\mathbb{R}^n}\left|\nabla\left(u^{p/2}\right)\right|^2 dx - \frac{\chi(p-1)}{p}\int_{\mathbb{R}^n} u^p\left(\nabla\cdot A\nabla v\right)dx.\end{aligned}$$

Applying Hölder's inequality, we find

$$\begin{aligned}&\|u^p\left(\nabla\cdot A\nabla v\right)\|_{L^1(\mathbb{R}^n)}\\ &\le \|u\|^p_{L^{p+1}(\mathbb{R}^n)}\|\nabla\cdot A\nabla v\|_{L^{p+1}(\mathbb{R}^n)} \le \|u\|^p_{L^{p+1}(\mathbb{R}^n)}\sum_{i,j=1,n}|a_{ij}|\,\|\partial_{ij}\mathbf{K}*u\|_{L^{p+1}(\mathbb{R}^n)}\\ &\le \|u\|^p_{L^{p+1}(\mathbb{R}^n)}\|A\|_{\max}\sum_{i,j=1,n}\|\partial_{ij}\mathbf{K}*u\|_{L^{p+1}(\mathbb{R}^n)}.\end{aligned}$$

Now, we recall the following Calderón–Zygmund inequality (See for instance [45, Section 6.4.2.]): For all $g \in L^q(\mathbb{R}^n)$, there exist a constant $C^{(q,n)}_{CZI} = C(q,n), 1 < q < \infty$, such that

$$\|\partial_{ij}\mathbf{K}*g\|_{L^q(\mathbb{R}^n)} \le C^{(q,n)}_{CZI}\|g\|_{L^q(\mathbb{R}^n)}\,,\ i,j = 1,2, \tag{6.11}$$

Taking $g = u$ and $q = p+1$, we deduce

$$\|u^p\left(\nabla\cdot A\nabla v\right)\|_{L^1(\mathbb{R}^n)} \le 4\|A\|_{\max}C^{(p+1,n)}_{CZI}\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}\,.$$

This leads to

$$\begin{aligned}&\frac{1}{(p-1)}\frac{d}{dt}\int_{\mathbb{R}^n}|u(x,t)|^p\,dx\\ &\le -\frac{4}{p}\int_{\mathbb{R}^n}\left|\nabla\left(u^{p/2}\right)\right|^2 dx + 4\chi\|A\|_{\max}C^{(p+1,n)}_{CZI}\int_{\mathbb{R}^n}u^{p+1}dx.\end{aligned} \tag{6.12}$$

Applying the Gagliardo-Nirenberg-Sobolev inequality, we obtain that for any $\frac{n}{2} \le p+1 \le \frac{pn}{n-2}$

$$\begin{aligned}\int_{\mathbb{R}^n}u^{p+1}dx &\le \|u\|_{L^{\frac{n}{2}}}\|u\|^p_{\frac{pn}{n-2}} = \|u\|_{L^{\frac{n}{2}}}\left\|u^{\frac{p}{2}}\right\|^2_{L^{\frac{2n}{n-2}}}\\ &\le C^2_{GNS}\|u\|_{L^{\frac{n}{2}}}\int_{\mathbb{R}^n}\left|\nabla\left(u^{p/2}\right)\right|^2 dx.\end{aligned} \tag{6.13}$$

Combining (6.12) and (6.13), we get for any $p \geq \max\{1, \frac{n}{2}-1\}$

$$\frac{1}{(p-1)}\frac{d}{dt}\int_{\mathbb{R}^n} |u(x,t)|^p\, dx \leq \left(4\chi \|A\|_{\max} C_{CZI}^{(p+1,n)} C_{GNS}^2 \|u\|_{L^{\frac{n}{2}}} - \frac{4}{p}\right)\int_{\mathbb{R}^n} \left|\nabla\sqrt{u}\right|^2 dx. \tag{6.14}$$

Notice that for $p = \frac{n}{2}$ in (6.14), the inequality $4\chi \|A\|_{\max} C_{CZI}^{(n/2+1,n)} C_{GNS}^2 \|u_0\|_{L^{\frac{n}{2}}} - \frac{8}{n} \leq 0$ implies that $\|u\|_{L^{\frac{n}{2}}}$ decreases for $t \in (0, T_{\max})$. As a consequence the condition

$$\|u_0\|_{L^{\frac{n}{2}}} \leq \frac{1}{\chi \|A\|_{\max} C_{GNS}^2} \min\left\{\frac{2}{nC_{CZI}^{(n/2+1,n)}}, \frac{1}{pC_{CZI}^{(p+1,n)}}\right\} =: \delta(p,n),$$

for $p \geq \max\{1, \frac{n}{2}-1\}$ implies that the function $\int_{\mathbb{R}^n} |u(x,t)|^p\, dx$ decreases for $t \in (0, T_{\max})$.
We fix any $q > n$, and let $\delta := \delta(q,n)$. Then, assuming that $\|u_0\|_{L^{\frac{n}{2}}} \leq \delta$, we obtain from (6.14) that there exists $c_1 > 0$ such that

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq c_1 \text{ for all } t \in (0, T_{\max}). \tag{6.15}$$

We recall now the following $L^q - L^p$ estimates of heat semigroup $e^{t\Delta}$. For any $1 \leq q \leq p \leq \infty$, there holds

$$\left\|e^{t\Delta} f\right\|_{L^p(\mathbb{R}^n)} \leq (4\pi t)^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^q(\mathbb{R}^n)}, \tag{6.16}$$

$$\left\|\nabla \cdot e^{t\Delta} F\right\|_{L^p(\mathbb{R}^n)} \leq C t^{-\frac{1}{2}+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|F\|_{L^q(\mathbb{R}^n)}, \tag{6.17}$$

where $C = C(p,q,n)$ is a constant depending only on p, q and n. These inequalities are a consequences of Young's inequality for the convolution (For example, see [45, Subsection 4.1.2. p. 145]).

Let us define

$$N(T) := \sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}, \text{ for } T \in (0, T_{\max}).$$

Using the Duhamel integral equation, we get

$$u(t) = e^{(t-t_0)\Delta} u(t_0) - \chi \int_{t_0}^{t} \nabla \cdot e^{(t-s)\Delta} \left(u(s) A \nabla v(s)\right) ds,$$

with

$$t_0 = \begin{cases} t-1, & \text{if } t \geq 1, \\ 0, & \text{if } 0 \leq t \leq 1. \end{cases}$$

By (6.16) and (6.17), we have that

$$\begin{aligned}
&\|u(x,t)\|_{L^\infty(\mathbb{R}^n)} \\
&\le \left\|e^{(t-t_0)\Delta}u(t_0)\right\|_{L^\infty(\mathbb{R}^n)} + \chi \left\|\int_{t_0}^{t} \nabla\cdot e^{(t-s)\Delta}\left(u(s)A\nabla v(s)\right) ds\right\|_{L^\infty(\mathbb{R}^n)} \\
&\le (4\pi(t-t_0))^{\frac{-n}{2q}} \|u(t_0)\|_{L^q(\mathbb{R}^n)} \\
&+ c_2 \int_{t_0}^{t} (t-s)^{-\frac{1}{2}-\frac{n}{2q}} \left\| |u(s)A\nabla v(s)| \right\|_{L^q(\mathbb{R}^n)} ds \\
&\le (4\pi(t-t_0))^{\frac{-n}{2q}} \|u(t_0)\|_{L^q(\mathbb{R}^n)} \\
&+ c_2 \|A\|_{\max} \int_{t_0}^{t} (t-s)^{-\frac{1}{2}-\frac{n}{2q}} \|u(s)\|_{L^q(\mathbb{R}^n)} \|\nabla v(s)\|_{L^\infty(\mathbb{R}^n)} ds,
\end{aligned}$$

for all $t \in (0,T)$. Notice that for any $\gamma > 0$, we have that

$$\begin{aligned}
&|\nabla v(x,s)| \\
&= |\nabla \mathbf{K}(x) * u(x,s)| \\
&= \left| \frac{-1}{n\,|B_1(0)|} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} u(y,s)dy \right| \\
&\le \frac{1}{n\,|B_1(0)|} \left(\int_{|x-y|\le\gamma} \frac{u(y,s)}{|x-y|^{n-1}} dy + \int_{|x-y|>\gamma} \frac{u(y,s)}{|x-y|^{n-1}} dy \right) \\
&\le \frac{\|u(x,s)\|_{L^\infty(\mathbb{R}^n)}}{n\,|B_1(0)|} \int_{|z|\le\gamma} |z|^{1-n} dz + \frac{\gamma^{1-n}\|u(x,s)\|_{L^1(\mathbb{R}^n)}}{n\,|B_1(0)|} \\
&= \gamma \|u(x,s)\|_{L^\infty(\mathbb{R}^n)} + \frac{\gamma^{1-n}M}{n\,|B_1(0)|}. \qquad (6.18)
\end{aligned}$$

Therefore, from (6.15) and (6.18)

$$\begin{aligned}
&\|u(x,t)\|_{L^\infty(\mathbb{R}^n)} \\
&\le c_1(4\pi)^{\frac{-n}{2q}} \\
&+ c_1c_2\|A\|_{\max}\left(\gamma N(T) + \frac{\gamma^{1-n}M}{n\,|B_1(0)|}\right) \int_{t_0}^{t} (t-s)^{-\frac{1}{2}-\frac{n}{2q}} ds, \qquad (6.19)
\end{aligned}$$

for all $t \in (0,T)$. Note that

$$\begin{aligned}
\int_{t_0}^{t} (t-s)^{-\frac{1}{2}-\frac{n}{2q}} ds &= \int_0^{t-t_0} \tau^{-\frac{1}{2}-\frac{n}{2q}} d\tau \\
&\le \int_0^1 \tau^{-\frac{1}{2}-\frac{n}{2q}} d\tau \\
&= \frac{2q}{q-n}. \qquad (6.20)
\end{aligned}$$

Taking

$$\frac{2qc_1c_2\|A\|_{\max}\gamma}{q-n} = \frac{1}{2},$$

we conclude from (6.19) and (6.20) that there exists a constant

$$c_3 := c_1(4\pi)^{\frac{-n}{2q}} + c_1 c_2 \|A\|_{\max} \frac{\gamma^{1-n} M}{n\,|B_1(0)|} > 0,$$

such that

$$N(T) \leq \frac{1}{2} N(T) + c_3 \text{ for all } T < T_{\max},$$

and hence

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq 2c_3 \text{ for all } t \in (0, T_{\max}),$$

as $T \in (0, T_{\max})$ was arbitrary. Taking into account the extensibility criterion in Preposition 53, the last inequality implies the global existence. ■

Remark 57 *Applying the inequality that compares the $L^{\frac{n}{2}}$-norm, the mass M, and the second moment m_0 of a non-negative function u_0 (See [11, Remark 2.6]):*

$$\|u_0\|_{L^{\frac{n}{2}}} \geq C_n M \left(\frac{M}{m_0}\right)^{\frac{n-2}{2}}, \tag{6.21}$$

where $C_n = C(n)$ is a constant depending only on n, we find that the condition (6.4) in Theorem 54 implies:

$$\|u_0\|_{L^{\frac{n}{2}}} \geq C_n (C_{Bl})^{\frac{2-n}{2}}.$$

Conversely, the smallness assumption on $\|u_0\|_{L^{\frac{n}{2}}}$ in Theorem 56 implies:

$$m_0 \geq \left(\frac{C_n}{\delta}\right)^{\frac{2}{n-2}} M^{\frac{n}{n-2}},$$

which shows the compatibility of both results.

Chapter 7

Remarks on Keller-Segel models describing Cell Aggregation with Obstacle Interference

Abstract

In this chapter, we investigate the effects of topographical obstacles during chemotaxis. Our approach involves modifying the Keller-Segel model by incorporating a spatially dependent coefficient of chemotaxis. Through our analysis, we demonstrate that this coefficient plays a crucial role in preventing blow-up phenomena in cell concentration. The research discussed in this chapter has been submitted for publication and is currently under review at the time of this thesis submission.

Directed, single-cell migration is driven by external guidance cues, such as chemical, electrical, temperature, stiffness, and topographical gradients (cf. [27, 22, 30, 71, 72, 56]). Natural cell environments often exhibit several such cues simultaneously. In the human body, processes occurring in multicue environments include immune response, cancer metastasis, and tissue regeneration. As of yet, it is unclear how external guidance cues relate to each other for various cell types and environments. Cells may ignore certain stimuli in favor of other cues or different cues might add up in affecting cell movement. In particular, contemporary research has illuminated the intricate relationship between chemotaxis and topography, shedding light on the influence of topographical cues on cellular chemotactic responses. Investigations have unveiled that the impact of topographical cues persists throughout cellular chemotaxis, with studies indicating that the topographical cue conserves its significance, contributing to the overall chemotactic effect (cf. [90]).

In this chapter, we focus on conditions that predict or prevent cell aggregation when obstacles interfere during the process. To this end, we propose to study the Keller-Segel-type model in a bounded domain $\Omega \subset \mathbb{R}^n$ given by:

$$\begin{aligned} \tfrac{\partial u}{\partial t}(x,t) &= \operatorname{div}\left[\nabla u(x,t) - \chi(x) u(x,t) \nabla v(x,t)\right], && t>0, x\in\Omega, \\ -\Delta v(x,t) &= u(x,t), v(x,t) = (K_n * u)(x,t), && t>0, x\in\Omega, \\ u(x,0) &= u_0(x), && x\in\Omega. \end{aligned} \tag{7.1}$$

Here K_n is the fundamental solution of the n-dimensional Laplacian, namely, $K_n(x) := |x|^{2-n} / (\sigma_n(n-2)), x \in \mathbb{R}^n, x \neq 0,\ n \geq 3$, where σ_n is the area of the unit sphere S^{n-1} and $K_2(x) = -\frac{1}{2\pi} \log |x|, x \in \mathbb{R}^2, x \neq 0$.

This model is complemented by the nonlinear no-flux condition:

$$\frac{\partial u(x,t)}{\partial n} - \chi(x) u(x,t) \frac{\partial v(x,t)}{\partial n} = 0, \tag{7.2}$$

where n denotes the outward unit normal vector to the $C^{1+\varepsilon}$ ($\varepsilon > 0$) boundary $\partial\Omega$. Here, u represents the cell density, v denotes the chemoattractant, and $\chi(x)$ represents the chemical response influenced by a topographical cue.

The initial-boundary value problem is supplemented with the initial condition:

$$u(x,0) = u_0(x) \geq 0. \tag{7.3}$$

The moment and mass be defined by $m(t) := \int u(x,t) |x|^2 \, dx$ and $M := \int u_0 dx = \int u(x,t) dx$

For any arbitrary bounded smooth domains in $\mathbb{R}^2$ or in $\mathbb{R}^3$, the local-in-time existence of solutions in $L^2(\Omega)$ can be deduced from the proof provided in Theorem 1 of the reference [7]. Although this Theorem specifically addresses the scenario where $\chi(x)$ is constant, its existence argument seemslessly extends to our situation by assuming $\chi(x) \in L^\infty(\Omega)$. This argument relies on a standard application of the Schauder fixed-point theorem within an appropriate space of vector-valued functions in $L^2(\Omega)$.

Definition 58 (Weak solution) *In the context of the problem (7.1)-(7.3) defined on $\Omega \times (0,T)$, weak $H^1(\Omega)$ solutions are understood as functions $u \in L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))$ which satisfy, for every test function $\eta \in H^1(\Omega \times (0,T))$ and for a.e. $t \in (0,T)$, the integral identity*

$$\int_\Omega u(x,t)\eta(x,t)dx - \int_0^t \int_\Omega u\eta_t + \int_0^t \int_\Omega (\nabla u + \chi(x) u \nabla v) \cdot \nabla \eta = \int_\Omega u_0(x)\eta(x,0)dx.$$

Moreover, we require that for a.e. $t \in (0,T), v(\cdot,t)$ is a weak solution of (7.1) with

$$v \in H^1(\Omega) \text{ with } \quad v = K_n * u.$$

Theorem 59 (Local Existence) *Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary of class $C^{1+\varepsilon}$, where $\varepsilon > 0$.*

(i) For dimension $n = 2$ or $n = 3$, and initial data $0 \leq u_0 \in L^2(\Omega)$, there exists $T = T(|u_0|_2)$ such that the problem (7.1)-(7.2) admits a unique weak solution $u \in L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))$. Additionally, $u_t \in L^2((0,T); H^{-1}(\Omega))$, $u(x,t) \geq 0$ for almost every $x \in \Omega$ and $t \geq 0$, and $\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx$.

(ii) For dimension $n \geq 2$ and $0 \leq u_0 \in L^p(\Omega)$ with $p > n/2$, there exists $T = T\left(p, |u_0|_p\right) > 0$ and a weak solution u such that $u \in L^\infty((0,T); L^p(\Omega))$ and $u^{p/2} \in L^2((0,T); H^1(\Omega))$.

These solutions are unique when $p > n$, and regular when $p > n/2$ in the sense that $u \in L^\infty_{loc}((0,T); L^\infty(\Omega))$.

Proof. The proof follows the same argument of [7, Theorem 1, Proposition 1] with minor modifications. ■

7.1 The role of topography during cell aggregation

The results in this section are motivated by the restriction to the case when $\chi(x)$ is a *radially increasing* function, which are functions $\chi : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\chi(x) \geq \chi(y) \quad \text{if} \quad |x| > |y| \,. \tag{7.4}$$

This type of function can be constructed by selecting an increasing function $f : \mathbb{R} \to \mathbb{R}^+$ and defining $\chi(x) := f(|x|^2)$. Specific examples are $\chi(x) = \frac{|x|^2}{1+|x|^2} + 1$ and $\chi(x) = \arctan |x|^2 + 1$.

Graphically, a $2D$ radially increasing function $\chi(x) := f(|x|^2)$ would appear as a bowl-shaped surface that is rotationally symmetric around the z-axis. Thus, the contour lines on the surface are concentric circles.

We call Ω a star-shaped domain if there exists $x_0 \in \mathbb{R}^n$ such that

$$(x - x_0) \cdot \nu \geq 0 \text{ for all } x \in \partial\Omega,$$

where ν is the unit outward normal to $\partial\Omega$ at x, cf. [73].

Theorem 60 (Blow-up in dimension two) *Let $\Omega \subset \mathbb{R}^2$ be a star-shaped domain respect to $0 \in \Omega$ and $\chi : \mathbb{R}^2 \to \mathbb{R}^+$ be a positive smooth function that increases radially; that is, satisfying the monotonicity condition (7.4). If the initial data satisfies $\int_\Omega u_0 dx =: M > \frac{4\pi}{\chi(\mathbf{0})}$, then the system (7.1)-(7.3) does not have global solutions.*

Proof. To simplify, we provide a formal argument demonstrating that the second moment $\int_\Omega u|x|^2\,dx$ becomes negative in a finite amount of time. The computations below can be justified by writing the integral version of the corresponding differential inequalities.
First, we observe that the cell-density u satisfies:

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u|x|^2\,dx \\
&= -2\int_\Omega \nabla u \cdot x\,dx + 2\int_\Omega x \cdot (u\chi(x)\nabla(K_2 * u))\,dx \\
&= -2\int_{\partial\Omega} u(x\cdot\nu)\,dx + 4\int_\Omega u\,dx + 2\int_\Omega x \cdot u\chi(x)\int_\Omega \frac{-1}{2\pi}\frac{x-y}{|x-y|^2}u\,dy\,dx.
\end{aligned}$$

Since Ω is a star-shaped domain respecto 0, we have $x \cdot \nu \geq 0$ on $\partial\Omega$, thus

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u|x|^2\,dx &\leq 4\int_\Omega u\,dx + 2\int_\Omega x \cdot u\chi(x)\int_\Omega \frac{-1}{2\pi}\frac{x-y}{|x-y|^2}u\,dy\,dx \\
&= 4\int_\Omega u_0\,dx + 2\int_\Omega x \cdot u\chi(x)\int_\Omega \frac{-1}{2\pi}\frac{x-y}{|x-y|^2}u\,dy\,dx \\
&= 4M - \frac{1}{\pi}I.
\end{aligned}$$

Next, we interchange x and y in the integral I to get

$$I = -\frac{1}{2}\int_{\Omega\times\Omega}\left(y\cdot\chi(y)\frac{x-y}{|x-y|^2}u(x,t)u(y,t)dy\right)dx,$$

and hence,

$$\frac{d}{dt}\int_{\Omega}u|\mathbf{x}|^2\,dx = 4M - \frac{1}{2\pi}\int_{\Omega\times\Omega}[\chi(x)x-\chi(y)y]\cdot\frac{x-y}{|x-y|^2}u(x,t)u(y,t)dydx.$$

A main difficulty arising at this point lies in estimating the last integral. To address this, we observe that

$$2[\chi(x)x-\chi(y)y]\cdot\frac{x-y}{|x-y|^2} = \chi(x)+\chi(y)+\frac{|x|^2-|y|^2}{|x-y|^2}[\chi(x)-\chi(y)].$$

which can be straighforward verified by expanding and comparing the expressions arising on each side of the equivalent identity

$$2[\chi(x)x-\chi(y)y]\cdot(x-y) = (\chi(x)+\chi(y))\,|x-y|^2+\left(|x|^2-|y|^2\right)[\chi(x)-\chi(y)].$$

Next, we apply the monoticity property (7.4) to obtain

$$[\chi(x)x-\chi(y)y]\cdot\frac{x-y}{|x-y|^2}\geq\frac{1}{2}\chi(x)+\frac{1}{2}\chi(y)\geq\chi(\mathbf{0}),$$

leading to the key estimate

$$\begin{aligned}\frac{d}{dt}\int_{\Omega}u|x|^2\,dx &\leq 4M-\frac{\chi(\mathbf{0})}{2\pi}\int_{\Omega\times\Omega}u(x,t)u(y,t)dydx\\ &= 4M-\frac{\chi(\mathbf{0})}{2\pi}M^2.\end{aligned}$$

Considering the assumption $\chi(0)>0$, we conclude that the second moment becomes negative in a finite amount of time if

$$M>\frac{8\pi}{\chi(\mathbf{0})},$$

which is absurd since u remains nonnegative. ■

Theorem 61 (Blow-up in dimension $n\geq 3$) *Let $\Omega\subset\mathbb{R}^n$ be a star-shaped domain with respect to $0\in\Omega$ and p a constant satisfying $2\leq p\leq n$. For $p=n$, assume that the initial data satisfies $\int_\Omega u_0dx =: M > \frac{2^n\sigma_n}{\chi}$, and for $2\leq p<n$ assume that*

$$\int_{\Omega}u_0\,|x|^2\,dx<\left(\frac{\chi}{2^{(p+n)/2}\sigma_n}\right)^{2/(n-p)}M^{(n-p+2)/(n-p)}.$$

Then the system

$$\begin{aligned}\partial_t u = \Delta u-\chi\nabla\cdot\left(|x|^{p-2}\,u\nabla v\right), &\qquad x\in\Omega, t>0,\\ -\Delta v = u,\ \ v(x,t)=\tfrac{1}{(n-2)\sigma_n}\textstyle\int_\Omega u(y,t)\,|x-y|^{2-n}\,dy, &\qquad x\in\Omega, t>0,\\ u(x,0)=u_0(x)\geq 0, &\qquad x\in\Omega,\end{aligned}$$

does not have global solutions.

Proof. We apply the moments' technique follows.

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u\,|x|^2\,dx &= -2\int_\Omega x\cdot\nabla u dx + 2\chi\int_\Omega x\cdot\left(|x|^{p-2}\,u\nabla v\right)dx \\
&\le 4\int_\Omega u dx + 2\chi\int_\Omega x\cdot\left(|x|^{p-2}\,u\nabla(K_n * u)\right)dx \\
&= 4\int_\Omega u_0 dx - \frac{2\chi}{\sigma_n}\int_\Omega x\cdot|x|^{p-2}\,u\left(\int_\Omega \frac{x-y}{|x-y|^n}u(y,t)dy\right)dx \\
&= 4M - \frac{2\chi}{\sigma_n}\int_{\Omega\times\Omega}\left(x\,|x|^{p-2}\cdot\frac{x-y}{|x-y|^n}u(x,t)u(y,t)dy\right)dx \\
&= 4M - \frac{2\chi}{\sigma_n}I.
\end{aligned}$$

We interchange x and y in the integral I to get

$$I = -\frac{1}{2}\int_{\Omega\times\Omega}\left(y\,|y|^{p-2}\cdot\frac{x-y}{|x-y|^n}u(x,t)u(y,t)dy\right)dx.$$

Thus

$$\frac{d}{dt}\int_\Omega u\,|x|^2\,dx \le 4M - \frac{\chi}{\sigma_n}\int_{\Omega\times\Omega}\left(|x|^{p-2}\,x - |y|^{p-2}\,y\right)\cdot\frac{x-y}{|x-y|^n}u(x,t)u(y,t)dydx.$$

Using the inequality (cf. [68])

$$\left(|x|^{p-2}\,x - |y|^{p-2}\,y\right)\cdot(x-y) \ge 2^{2-p}\,|x-y|^p\,,$$

for all $x, y \in \mathbb{R}^n$, and $p \ge 2$, we get

$$\frac{d}{dt}\int_\Omega u\,|x|^2\,dx \le 4M - \frac{2^{2-p}\chi}{\sigma_n}\int_{\Omega\times\Omega}|x-y|^{p-n}\,u(x,t)u(y,t)dydx. \tag{7.5}$$

We consider separately now two cases: $p = n$ and $2 \le p < n$. Firstly, when $p = n$, we obtain from (7.5)

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u\,|x|^2\,dx &\le 4M - \frac{2^{2-n}\chi}{\sigma_n}\int_{\Omega\times\Omega}u(x,t)u(y,t)dydx \\
&= 4M - \frac{2^{2-n}\chi}{\sigma_n}M^2.
\end{aligned} \tag{7.6}$$

Then, we conclude that we have blow-up if $4 < 2^{2-n}\chi M/\sigma_n$ or equivalently

$$M > \frac{2^n\sigma_n}{\chi}.$$

For $2 \le p < n$, using Lemma 62 (See [12, Lemma 3.2.]), we estimate the last integral as

$$\int_{\Omega\times\Omega}|x-y|^{p-n}\,u(x,t)u(y,t)dydx \ge M^{2+(n-p)/2}\,(2m(t))^{(p-n)/2}\,.$$

Thus, we get

$$\frac{d}{dt}m(t) \leq 4M - \frac{2^{2-(p+n)/2}\chi}{\sigma_n} M^{2+(n-p)/2} (m(t))^{(p-n)/2}$$
$$= f(m(t)),$$

and we let $f(m(0)) < 0$, or equivalently

$$m(0) < \left(\frac{\chi}{2^{(p+n)/2}\sigma_n}\right)^{2/(n-p)} M^{(n-p+2)/(n-p)} =: CM^{(n-p+2)/(n-p)}. \quad (7.7)$$

Taking into account that the condition (7.7) implies that $m(t)$ is decreasing for t small enough and the fact that f is an increasing function of m, we conclude that the right hand side is always negative and bounded away from $f(m(0)) < 0$. It follows from (7.7) that $m(t)$ will become negative a finite amount of time. On the other hand, $m(t)$ remains always positive due the nonnegativity of the variable u. This contradiction implies $T_{\max} < \infty$. ■

Lemma 62 *Let for a density* $0 \leq u \in L^1(\mathbb{R}^n, (1+|x|^2)dx)$ *the moment and mass be defined by* $m = \int u(x)|x|^2 dx$ *and* $M = \int u(x)dx$*, respectively. Then for the integral*

$$J = \int_{\mathbb{R}^n \times \mathbb{R}^n} u(x)u(y)|x-y|^{p-n} dydx,$$

with $p \leq n$*, the inequality*

$$M^{2+(n-p)/2} \leq J(2m)^{(n-p)/2}, \quad (7.8)$$

holds

Proof. Using the Holder inequality, we have that

$$M^2 = \int_{\mathbb{R}^n \times \mathbb{R}^n} u(x)u(y)dxdy$$
$$\leq \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} u(x)u(y)|x-y|^2 dxdy\right)^{1-\frac{2}{n-p+2}}$$
$$\left(\int_{\mathbb{R}^n \times \mathbb{R}^n} u(x)u(y)|x-y|^{p-n} dxdy\right)^{\frac{2}{n-p+2}}$$
$$= \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} u(x)u(y)\left(|x|^2+|y|^2-2x\cdot y\right) dxdy\right)^{1-\frac{2}{n-p+2}} J^{\frac{2}{n-p+2}}$$
$$\leq \left(2Mm - 2\left|\int_{\mathbb{R}^n} xu(x)dx\right|^2\right)^{1-\frac{2}{n-p+2}} J^{\frac{2}{n-p+2}}.$$

which implies (7.8). ■

7.2 Global existence for the case $\chi(x) \propto |x|^{n-2}$

Throughout this section, we assume that

$$\Omega = \{x \in \mathbb{R}^n | \ |x| < L\}, n = 2, 3, 4, 5, \dots, 0 < L < \infty.$$

We discuss in this section the global existence of radially symmetric densities $u(x,t) = u(|x|,t)$ in the ball $B(0,R) \subset \mathbb{R}^n$. satisfying the system

$$\begin{array}{ll} \partial_t u = \Delta u - \chi \nabla \cdot (|x|^{n-2} u \nabla v), & x \in \Omega, t > 0, \\ -\Delta v = u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{array} \tag{7.9}$$

The following proposition and lemmas are shown for solutions to the case when $\chi(x)$ remains constants for every x in [63]. However, by using a similar argument as the one in [63], we can show the following lemma for solutions to (7.9). Hence, here we omit the proofs.

Proposition 63 . *The system (7.9) has the unique classical solution u in $\Omega \times (0, T_{\max})$. Moreover, u is positive in $\bar{\Omega} \times (0, T_{\max})$.*

Then, the maximal existence time $T_{\max}$ of the classical solution is positive or infinite.

Lemma 64 *Let u be a solution to (7.9). If $T_{\max} < \infty$, then u satisfies that*

$$\lim_{t \to T_{\max}} \|u(\cdot,t)\|_\infty = \infty.$$

Lemma 65 *Let u be a solution to (7.9) and $u_0 \in L^1(\Omega) \cap L^\infty(\Omega)$. Suppose that*

$$\sup_{0<t<T_{\max}} \|\nabla v(\cdot,t)\|_\infty < \infty.$$

Then it holds that

$$\sup_{0<t<T_{\max}} \|u(\cdot,t)\|_\infty < \infty.$$

We assume that u_0 is radial and that Ω is a bounded open ball.

Theorem 66 *Let $\Omega = \{x \in \mathbb{R}^n | \ |x| < L\}, 0 < L < \infty, n \geq 2$ and $u_0 \in L^1(\Omega) \cap L^\infty(\Omega)$. Assume that $\int_\Omega u_0 dx < \frac{2n\sigma_n}{\chi}$ then, the solution u to (7.9) exists globally in time and satisfies $sup_{t>0}\|u(\cdot,t)\|_\infty < \infty$.*

Proof. We define the cumulative mass $M(r,t)$

$$M(r,t) := \int_{B(0,r)} u(x,t)dx = \sigma_n \int_0^r u(\rho,t)\rho^{n-1}d\rho.$$

It is follows that $M(r,t)$ satisfies

$$\begin{array}{ll} M_t = M_{rr} - (n-1)r^{-1}M_r + \chi\sigma_n^{-1}r^{-1}MM_r, & 0 < r < L, 0 < t < T_{\max}, \\ M(0,t) = 0, M(L,t) = \theta, & 0 < t < T_{\max}, \\ M(r,0) = \sigma_n \int_0^r u_0(\rho)\rho^{n-1}d\rho, & 0 \leq r \leq L. \end{array} \tag{7.10}$$

Next, consider the following ODE

$$0 = \overline{M}_{rr} - (n-1)r^{-1}\overline{M}_r + \chi\sigma_n^{-1}r^{-1}\overline{M}\overline{M}_r,$$

Then

$$\begin{aligned} 0 &= \int_0^r \rho\overline{M}_{\rho\rho}d\rho - (n-1)\int_0^r \overline{M}_\rho d\rho + \chi\sigma_n^{-1}\int_0^r \overline{M}\overline{M}_\rho d\rho \\ &= r\overline{M}_r - \overline{M} - (n-1)\overline{M} + \chi\left(2\sigma_n\right)^{-1}\overline{M}^2 \\ &= r\overline{M}_r - n\overline{M} + \chi\left(2\sigma_n\right)^{-1}\overline{M}^2. \end{aligned}$$

or equivalently

$$r\frac{d\overline{M}}{dr} = n\overline{M} - \chi\left(2\sigma_n\right)^{-1}\overline{M}^2 = \overline{M}\left(n - \chi\left(2\sigma_n\right)^{-1}\overline{M}\right).$$

Separation of variables leads to

$$\overline{M}(r) = \frac{2n\sigma_n}{\chi}\frac{kr^n}{1+kr^n} < \frac{2n\sigma_n}{\chi}.$$

where k is the constant of integration. We note that k can be choosen sufficiently large such that

$$\theta < \overline{M}(L) \text{ and } M(r,0) \le Cr^n \le \overline{M}(r) \text{ for } 0 \le r \le L,$$

where $C = n^{-1}\sigma_n \left\|u_0\right\|_{L^\infty}$. By the comparison theorem

$$M(r,t) \le \overline{M}(r) \text{ for } 0 \le r \le L, 0 \le t < T_{\max}.$$

Consequently,

$$\begin{aligned} |\nabla v(x,t)| = |\partial_r v(r,t)| &= \sigma_n^{-1}r^{1-n}M(r,t) \\ &\le \sigma_n^{-1}r^{1-n}\overline{M}(r) \le \frac{2n\sigma_n}{\chi}\frac{kr}{1+kr^n} \le \frac{2n\sigma_n kL}{\chi}. \end{aligned}$$

From this, Lemmas 64 and 65, we get this theorem. ■

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